POINCARÉ SERIES OF SOME PURE AND MIXED TRACE ALGEBRAS OF TWO GENERIC MATRICES

DRAGOMIR Ž. ĐOKOVIĆ

ABSTRACT. We work over a field K of characteristic zero. The Poincaré series for the algebra $C_{n,2}$ of GL_n-invariants and the algebra $T_{n,2}$ of GL_n-concomitants of two generic $n \times n$ matrices x and y are presented for $n \leq 6$. Both simply graded and bigraded cases are included. The cases $n \leq 4$ were known previously. If $n = 5$ or 6, we show that $C_{n,2}$ has no bigraded system of parameters.

For the algebra $C_{4,2}$ we construct a minimal set of generators, each of the form $\mathrm{tr}\,(w(x, y))$ where w is a word in two letters, and give an application to Specht's theorem on unitary similarity of two complex matrices. We also construct a minimal set of generators for the algebra $C_{5,2}$. It consists of 173 bihomogeneous polynomials. Five of them are symmetric (i.e., satisfy $f(x, y) = f(y, x)$), four are skew-symmetric and the remaining 164 generators occur in pairs $\{f(x, y), f(y, x)\}$.

By identifying the space M_n^2 of pairs of $n \times n$ matrices with $M_n \otimes K^2$, we extend the action of GL_n to $\mathrm{GL}_n \times \mathrm{GL}_2$. For $n \leq 5$, we compute the Poincaré series for the polynomial invariants of this action when restricted to the subgroups $\mathrm{GL}_n \times \mathrm{SL}_2$ and $\mathrm{GL}_n \times \Delta_1$, where Δ_1 is the maximal torus of SL_2 consisting of diagonal matrices.

Five conjectures are proposed concerning the lowest term numerators and denominators of various Poincaré series mentioned above. Some heuristic formulas and open problems are indicated.

1. Introduction

Let K be a field of characteristic 0, $M_n = M_n(K)$ the K-algebra of $n \times n$ matrices over K, and d a positive integer. The general linear group $\mathrm{GL}_n = \mathrm{GL}_n(K)$ acts on the direct product M_n^d of d copies of M_n by simultaneous conjugation

$$a \cdot (x_1, \ldots, x_d) = (ax_1a^{-1}, \ldots, ax_da^{-1}).$$

This gives rise to an action of GL_n on the algebra $K[M_n^d]$ of polynomial functions on M_n^d. We shall view the entries of x_k, $k = 1, \ldots, d$, also as linear functions on M_n^d, thus providing a convenient system of coordinates on this space.

It is a well known fact, due to Procesi [15] and Razmyslov [16], that the algebra $C_{n,d} = K[M_n^d]^{\mathrm{GL}_n}$ of GL_n-invariants in $K[M_n^d]$ is generated by all traces

$$\mathrm{tr}\,(z_1z_2 \cdots z_k), \quad z_1, z_2, \ldots, z_k \in \{x_1, \ldots, x_d\}, \quad k \geq 1.$$

This assertion remains valid if one imposes the restriction $k \leq n^2$. For these facts and many other known properties of $C_{n,d}$ we refer the interested reader to one of the references [4, 6, 9] and the papers quoted there.

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Similarly, GL_n acts on the (noncommutative) algebra of polynomial maps $M_n^d \to M_n$. Its subalgebra consisting of GL_n-equivariant maps will be denoted by $T_{n,d}$. The algebra $C_{n,d}$ resp. $T_{n,d}$ is known as the *pure* resp. *mixed trace algebra* of d generic $n \times n$ matrices.

By assigning the degree $(1, 0, \ldots, 0)$ to the entries of the matrix x_1, the degree $(0, 1, 0, \ldots, 0)$ to the entries of x_2, etc., one obtains a $\mathbf{Z}^d$-gradation of the algebras $C_{n,d}$ and $T_{n,d}$. The total degree provides these algebras with the ordinary $\mathbf{Z}$-gradation. In this paper we are mostly interested in the case $d = 2$. Until further notice, we assume that this is the case and we set $x = x_1$ and $y = x_2$.

We shall now describe briefly the contents of each section. The simply graded and bigraded Poincaré series for both $C_{n,2}$ and $T_{n,2}$ have been explicitly computed for $n \leq 4$ (see [1, 20, 21, 22]).

In Section 2 resp. 3 we present the results of our computations giving the bigraded resp. simply graded Poincaré series for the algebras $C_{n,2}$ and $T_{n,2}$ for $n = 5, 6$. For the bigraded case see Theorem 2.2 and Tables 1 and 2. For the simply graded case see Tables 3 and 4, as well as Tables 9 and 11 in Appendices C and D, respectively. We also show (see Proposition 2.1) that the $\mathbf{Z}^2$-graded algebra $C_{n,2}$ has no bigraded system of parameters if $n = 5, 6$.

In Section 4 we construct a minimal set of generators of $C_{4,2}$ (see Theorem 4.2) which consists of 32 elements of the form $\mathrm{tr}\,(w(x, y))$, where $w(x, y)$ is a word in the matrices x and y. This problem has been already solved by Drensky and Sadikova [5], but their generators are not of this simple form. However, they claim that their choice is better suited for finding a presentation of $C_{4,2}$, which is apparently still an open problem.

We give an application to the problem of unitary similarity of two complex 4×4 matrices. Recall that Pearcy [14] showed more than 40 years ago that two 3×3 complex matrices a and b are unitarily similar iff the equalities $\mathrm{tr}\,(w(a, a^*)) = \mathrm{tr}\,(w(b, b^*))$ hold when w runs through his list of nine words. Sibirskiĭ [18] showed later that two of these nine words are redundant. We provide a similar test for unitary similarity of two 4×4 complex matrices a and b. Some of the words in our generating set occur in pairs $\{w_1, w_2\}$ such that $\mathrm{tr}\,(w_1(a, a^*))$ and $\mathrm{tr}\,(w_2(a, a^*))$ are complex conjugates of each other for any complex matrix a. Hence we can omit a word from any such pair. In this way we obtain a reduced list of only 20 words which suffices for testing the unitary similarity of 4×4 complex matrices (see Theorem 4.4).

In Section 5 we construct a minimal set of generators, $\mathcal{P}$, for the algebra $C_{5,2}$. It consists of 173 bihomogeneous polynomials.

In Section 6 we identify the space M_n^d with the tensor product $M_n \otimes K^d$ and extend the action of GL_n to $\mathrm{GL}_n \times \mathrm{GL}_d$ by letting GL_d act on K^d by multiplication. We denote by $C_{n,d}^{\#}$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n \times \mathrm{SL}_d$-invariant functions. In Table 7 we record the results of our computation of the Poincaré series of $C_{n,2}^{\#}$, $n \leq 5$.

In Section 7 we restrict the action of $\mathrm{GL}_n \times \mathrm{SL}_d$ to $\mathrm{GL}_n \times \Delta_{d-1}$, where Δ_{d-1} is the maximal torus of SL_d consisting of diagonal matrices. We denote by $C_{n,d}^{\bullet}$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n \times \Delta_{d-1}$-invariant functions. In Table 8 we record the Poincaré series for the algebras $C_{n,2}^{\bullet}$, $n \leq 5$.

In Section 8 we propose four conjectures concerning the lowest term numerators and denominators of the Poincaré series of $C_{n,2}$ and $T_{n,2}$, one more conjecture about the series for $C^{\#}_{n,2}$ and $C^{\bullet}_{n,2}$, and state an open problem.

The numerators of the bigraded Poincaré series of $C_{6,2}$ and $T_{6,2}$ have 1169 and 854 terms, respectively. For that reason we give them separately in Appendix A. By using the fact that these polynomials are symmetric in the two variables and that they satisfy a simple functional equation, we can write them in a compact form.

Appendix B gives some details concerning the verification that our expression for $P(C_{5,2}; s, t)$ agrees with Formanek's expansion in terms of Schur functions (see [1]).

Appendix C contains the Taylor expansions (listing all coefficients of degree less than 20) of $P(C_{n,2}; t)$ for $n \leq 6$ and for the closely related algebras $C_{n,2}(0)$, introduced in Section 5. We also tabulate some of these coefficients, make a couple of interesting observations, and indulge in some speculative thinking.

Appendix D treats in the same way the Taylor expansions of $P(T_{n,2}; t)$ for $n \leq 6$.

We warn the reader that several assertions and formulae that appear in the last two appendices are of hypothetical character and are included there only as a suggestion to the interested reader as deserving further consideration and study.

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2. Bigraded Poincaré series

Let $P(C_{n,2}; s, t)$ resp. $P(T_{n,2}; s, t)$ denote the bigraded Poincaré series of the pure resp. mixed trace algebra $C_{n,2}$ resp. $T_{n,2}$. It is known that these series are given by symmetric rational functions in the two variables s and t.

We can write these rational functions in lowest terms as

$$P(C_{n,2}; s, t) = \frac{N(C_{n,2}; s, t)}{D(C_{n,2}; s, t)} \tag{2.1}$$

and

$$P(T_{n,2}; s, t) = \frac{N(T_{n,2}; s, t)}{D(T_{n,2}; s, t)}, \tag{2.2}$$

where we normalize N and D by demanding that they both have constant term 1.

We recall that Van den Bergh [23, Proposition 5.1] has shown (in a more general setting) that these two rational functions can be written as fractions whose numerator is a polynomial with integer coefficients and the denominator is a product of terms $1 - u$ with u a monomial in s and t. He does not claim that the latter property holds if the fraction is in lowest terms. However, this is indeed true for $n \leq 6$.

The Poincaré series (2.1) and (2.2) have been computed by Teranishi [20, 21, 22] for $n \leq 4$. The formulae for the case $n = 4$ were computed independently by Berele and Stembridge [1]. They have also corrected several misprints in Teranishi's formulas in [21].

For the sake of completeness and the reader's convenience, we give explicit formulae for the numerators and denominators for all $n \leq 6$. For $n \leq 5$, the numerators $N(C_{n,2}; s, t)$ resp. $N(T_{n,2}; s, t)$ are given in Table 1 resp. 2. For $n = 6$, the numerators are huge and are given in Appendix A.

We remark that these numerators are symmetric polynomials in s and t, and that they also satisfy the functional equation

$$(st)^d N(C_{n,2}; s^{-1}, t^{-1}) = N(C_{n,2}; s, t),$$

where d is the degree of $N(C_{n,2}; s, t)$ as a polynomial in s. For that reason (and to save space) there is no need to write all the terms of $N(C_{n,2}; s, t)$. This remark also applies to the numerators $N(T_{n,2}; s, t)$.

Table 1: Numerators $N(C_{n,2}; s, t)$

$$
\begin{aligned}
&N(C_{1,2}; s,t) = N(C_{2,2}; s,t) = 1,\\
&N(C_{3,2}; s,t) = 1 - st + s^2t^2\,,\\
&N(C_{4,2}; s,t) =\\
&\left(1 - st + s^2t^2\right)\left(1 - st^2 - s^2t + s^2t^3 + s^3t^2 + s^2t^4 + 2\,s^3t^3 + s^4t^2 + s^3t^4\right.\\
&\left. + s^4t^3 - s^4t^5 - s^5t^4 + s^6t^6\right)\\
&= 1 - st - st^2 - s^2t + s^2t^2 + 2\,s^2t^3 + 2\,s^3t^2 + s^2t^4 + 2\,s^3t^3 + s^4t^2 - s^3t^4\\
&- s^4t^3 - s^3t^5 - 2\,s^4t^4 - s^5t^3 - \cdots - s^7t^7 + s^8t^8,\\
&N(C_{5,2}; s,t) =\\
&1 - st - 2\,st^2 - 2\,s^2t - st^3 - s^3t + 3\,s^2t^3 + 3\,s^3t^2 + 4\,s^2t^4 + 7\,s^3t^3 + 4\,s^4t^2\\
&+3\,s^2t^5 + 4\,s^3t^4 + 4\,s^4t^3 + 3\,s^5t^2 + s^2t^6 - 2\,s^3t^5 - 5\,s^4t^4 - 2\,s^5t^3 + s^6t^2\\
&-5\,s^3t^6 - 13\,s^4t^5 - 13\,s^5t^4 - 5\,s^6t^3 - 4\,s^3t^7 - 10\,s^4t^6 - 11\,s^5t^5 - 10\,s^6t^4\\
&-4\,s^7t^3 - 2\,s^3t^8 - 2\,s^4t^7 + 3\,s^5t^6 + 3\,s^6t^5 - 2\,s^7t^4 - 2\,s^8t^3 + 5\,s^4t^8 + 18\,s^5t^7\\
&+27\,s^6t^6 + 18\,s^7t^5 + 5\,s^8t^4 + 4\,s^4t^9 + 18\,s^5t^8 + 29\,s^6t^7 + 29\,s^7t^6 + 18\,s^8t^5\\
&+4\,s^9t^4 + 2\,s^4t^{10} + 8\,s^5t^9 + 12\,s^6t^8 + 11\,s^7t^7 + 12\,s^8t^6 + 8\,s^9t^5 + 2\,s^{10}t^4\\
&-s^5t^{10} - 11\,s^6t^9 - 24\,s^7t^8 - 24\,s^8t^7 - 11\,s^9t^6 - s^{10}t^5 - 3\,s^5t^{11} - 17\,s^6t^{10}\\
&-40\,s^7t^9 - 47\,s^8t^8 - 40\,s^9t^7 - 17\,s^{10}t^6 - 3\,s^{11}t^5 - s^5t^{12} - 10\,s^6t^{11} - 24\,s^7t^{10}\\
&-33\,s^8t^9 - 33\,s^9t^8 - 24\,s^{10}t^7 - 10\,s^{11}t^6 - s^{12}t^5 - s^6t^{12} - s^7t^{11} + 9\,s^8t^{10}\\
&+13\,s^9t^9 + 9\,s^{10}t^8 - s^{11}t^7 - s^{12}t^6 + s^6t^{13} + 10\,s^7t^{12} + 34\,s^8t^{11} + 54\,s^9t^{10}\\
&+54\,s^{10}t^9 + 34\,s^{11}t^8 + 10\,s^{12}t^7 + s^{13}t^6 + 6\,s^7t^{13} + 27\,s^8t^{12} + 49\,s^9t^{11}\\
&+62\,s^{10}t^{10} + 49\,s^{11}t^9 + 27\,s^{12}t^8 + 6\,s^{13}t^7 + 5\,s^8t^{13} + 10\,s^9t^{12} + 13\,s^{10}t^{11}\\
&+13\,s^{11}t^{10} + 10\,s^{12}t^9 + 5\,s^{13}t^8 - 4\,s^8t^{14} - 19\,s^9t^{13} - 37\,s^{10}t^{12} - 45\,s^{11}t^{11}\\
&-37\,s^{12}t^{10} - 19\,s^{13}t^9 - 4\,s^{14}t^8 - 2\,s^8t^{15} - 20\,s^9t^{14} - 48\,s^{10}t^{13} - 72\,s^{11}t^{12}\\
&-72\,s^{12}t^{11} - 48\,s^{13}t^{10} - 20\,s^{14}t^9 - 2\,s^{15}t^8 - \cdots - s^{22}t^{22} + s^{23}t^{23}.
\end{aligned}
$$

One can easily verify that all coefficients of the extended numerators

$$
\begin{aligned}
N^*(C_{3,2}; s,t) &= (1+st)\,N(C_{3,2}; s,t),\\
N^*(C_{4,2}; s,t) &= (1+st)\left(1+st^2\right)\left(1+s^2t\right)N(C_{4,2}; s,t)
\end{aligned}
$$

are nonnegative integers. The corresponding extended denominators

$$
\begin{aligned}
D^*(C_{3,2}; s,t) &= (1+st)\,D(C_{3,2}; s,t),\\
D^*(C_{4,2}; s,t) &= (1+st)\left(1+st^2\right)\left(1+s^2t\right)D(C_{4,2}; s,t)
\end{aligned}
$$

can still be written as products of the binomials of the form $1-u$. Explicitly, we have

$$\begin{aligned}
N^*(C_{3,2};s,t) &= 1+s^3t^3,\\
D^*(C_{3,2};s,t) &= (1-s)(1-t)\cdot\left(1-s^2\right)(1-st)\left(1-t^2\right)\cdot\\
&\quad \left(1-s^3\right)\left(1-s^2t\right)\left(1-st^2\right)\left(1-t^3\right)\cdot\left(1-s^2t^2\right),\\
N^*(C_{4,2};s,t) &= 1+s^2t^3+s^3t^2+2\,s^3t^3+s^3t^4+s^4t^3+s^3t^5+2\,s^4t^4+s^5t^3\\
&\quad +s^3t^6+s^4t^5+s^5t^4+s^6t^3+s^4t^6+2\,s^5t^5+s^6t^4+2\,s^5t^6\\
&\quad +2\,s^6t^5+2\,s^6t^6+\cdots+s^9t^{10}+s^{10}t^9+s^{12}t^{12},\\
D^*(C_{4,2};s,t) &= (1-s)(1-t)\cdot\left(1-s^2\right)(1-st)\left(1-t^2\right)\cdot\\
&\quad \left(1-s^3\right)\left(1-s^2t\right)\left(1-st^2\right)\left(1-t^3\right)\cdot\\
&\quad \left(1-s^4\right)\left(1-s^3t\right)\left(1-s^2t^2\right)^2\left(1-st^3\right)\left(1-t^4\right)\cdot\\
&\quad \left(1-s^4t^2\right)\left(1-s^2t^4\right).
\end{aligned}$$

On the other hand, the Poincaré series $P(C_{5,2};s,t)$ and $P(C_{6,2};s,t)$ do not admit similar expressions because $N(C_{5,2};1,1)=N(C_{6,2};1,1)=0$.

Proposition 2.1. *The algebras $C_{5,2}$ and $C_{6,2}$ have no bigraded system of parameters.*

Proof. Assume that $C_{5,2}$ has a bigraded system of parameters, say $\{P_1,\ldots,P_{26}\}$. Since $C_{5,2}$ is a Cohen–Macauley algebra, it is a free graded module over the polynomial algebra $K[P_1,\ldots,P_{26}]$. Consequently, this module has a free bigraded basis, say $\{Q_1=1,\ Q_2,\ldots,Q_m\}$. It follows that

$$P(C_{5,2};s,t)=\frac{\sum_{j=1}^m s^{e'_j}t^{e''_j}}{\prod_{i=1}^{26}\left(1-s^{d'_i}t^{d''_i}\right)}=\frac{N(C_{5,2};s,t)}{D(C_{5,2};s,t)},$$

where (d'_i,d''_i) resp. (e'_j,e''_j) is the bidegree of P_i resp Q_j. As $N(C_{5,2};s,t)$ and $D(C_{5,2};s,t)$ are relatively prime, we deduce that

$$\sum_{j=1}^m s^{e'_j}t^{e''_j}=N(C_{5,2};s,t)R(s,t),$$

where $R(s,t)$ is a polynomial in s and t. By setting $s=t=1$ in this identity, we obtain the contradiction $m=0$ since $N(C_{5,2};1,1)=0$.

The proof for the algebra $C_{6,2}$ is similar. □

Table 2: Numerators $N(T_{n,2}; s,t)$

$$
\begin{aligned}
&N(T_{1,2}; s,t) = N(T_{2,2}; s,t) = N(T_{3,2}; s,t) = 1,\\
&N(T_{4,2}; s,t) = 1 + s^2t^2 + s^2t^3 + s^3t^2 + s^3t^3 + s^5t^5,\\
&N(T_{5,2}; s,t) =\\
&1 - st^2 - s^2t + s^2t^2 + 2\,s^2t^3 + 2\,s^3t^2 + 2\,s^2t^4 + 5\,s^3t^3 + 2\,s^4t^2 + s^2t^5 + 2\,s^3t^4\\
&+2\,s^4t^3 + s^5t^2 - s^3t^6 - 3\,s^4t^5 - 3\,s^5t^4 - s^6t^3 - s^3t^7 - s^4t^6 + s^5t^5 - s^6t^4 - s^7t^3\\
&+4\,s^5t^6 + 4\,s^6t^5 + s^4t^8 + 6\,s^5t^7 + 12\,s^6t^6 + 6\,s^7t^5 + s^8t^4 + 3\,s^5t^8 + 6\,s^6t^7\\
&+6\,s^7t^6 + 3\,s^8t^5 + s^6t^8 + s^7t^7 + s^8t^6 - s^5t^{10} - 5\,s^6t^9 - 9\,s^7t^8 - 9\,s^8t^7 - 5\,s^9t^6\\
&-s^{10}t^5 - 4\,s^6t^{10} - 9\,s^7t^9 - 7\,s^8t^8 - 9\,s^9t^7 - 4\,s^{10}t^6 - s^6t^{11} - 5\,s^7t^{10} - 5\,s^8t^9\\
&-5\,s^9t^8 - 5\,s^{10}t^7 - s^{11}t^6 + 3\,s^8t^{10} + 6\,s^9t^9 + 3\,s^{10}t^8 + 4\,s^8t^{11} + 6\,s^9t^{10}\\
&+6\,s^{10}t^9 + 4\,s^{11}t^8 + \cdots - s^{17}t^{18} - s^{18}t^{17} + s^{19}t^{19}.
\end{aligned}
$$

The denominators $D(C_{n,2}; s,t)$ and $D(T_{n,2}; s,t)$ are closely related to the product

$$
\Pi_n(s,t) = \prod_{i=1}^{n}(1-s^i)(1-t^i)\prod_{j=1}^{i-1}(1-s^{i-j}t^j)^{\min(i,n+1-i)}. \tag{2.3}
$$

Theorem 2.2. *For $n \le 5$, $N(C_{n,2}; s,t)$ and $N(T_{n,2}; s,t)$ are given by Tables 1 and 2, respectively. For $n = 6$ they are given in Appendix A. For $n \le 6$, $D(C_{n,2}; s,t)$ and $D(T_{n,2}; s,t)$ are given by*

$$
\begin{aligned}
&D(C_{n,2}; s,t) = \Pi_n(s,t), \quad n \le 5;\\
&D(C_{6,2}; s,t) = (1-st)\Pi_6(s,t);\\
&D(T_{n,2}; s,t) = (1+s+\cdots+s^{n-1})^{-1}(1+t+\cdots+t^{n-1})^{-1}D(C_n,2; s,t).
\end{aligned}
$$

For instance, we have

$$
\begin{aligned}
D(C_{5,2}; s,t) \;=\;& (1-s)\,(1-t)\cdot\left(1-s^2\right)(1-st)^2\left(1-t^2\right)\cdot\\
&\left(1-s^3\right)\left(1-s^2t\right)^3\left(1-st^2\right)^3\left(1-t^3\right)\cdot\\
&\left(1-s^4\right)\left(1-s^3t\right)^2\left(1-s^2t^2\right)^2\left(1-st^3\right)^2\left(1-t^4\right)\cdot\\
&\left(1-s^5\right)\left(1-s^4t\right)\left(1-s^3t^2\right)\left(1-s^2t^3\right)\left(1-st^4\right)\left(1-t^5\right),\\
D(T_{5,2}; s,t) \;=\;& (1-s)^2(1-t)^2\cdot\left(1-s^2\right)(1-st)^2\left(1-t^2\right)\cdot\\
&\left(1-s^3\right)\left(1-s^2t\right)^3\left(1-st^2\right)^3\left(1-t^3\right)\cdot\\
&\left(1-s^4\right)\left(1-s^3t\right)^2\left(1-s^2t^2\right)^2\left(1-st^3\right)^2\left(1-t^4\right)\cdot\\
&\left(1-s^4t\right)\left(1-s^3t^2\right)\left(1-s^2t^3\right)\left(1-st^4\right).
\end{aligned}
$$

Proof. By using the well known Molien–Weyl formula (see [2]), we have:

$$
\begin{aligned}
P(C_{n,2}; s,t) \;=\;& \frac{1}{(1-s)^n(1-t)^n}\cdot\frac{1}{(2\pi i)^{n-1}}\cdot\\
&\int_{|x_1|=1}\cdots\int_{|x_{n-1}|=1}\prod_{1\le k\le r\le n-1}\frac{1-x_kx_{k+1}\cdots x_r}{\varphi_{k,r}}\frac{\mathrm{d}x_{n-1}}{x_{n-1}}\cdots\frac{\mathrm{d}x_1}{x_1},
\end{aligned}
$$

where

$$\begin{aligned}\varphi_{k,r} &= (1 - s x_k x_{k+1} \cdots x_r)(1 - t x_k x_{k+1} \cdots x_r) \cdot \\ &\quad (1 - s(x_k x_{k+1} \cdots x_r)^{-1})(1 - t(x_k x_{k+1} \cdots x_r)^{-1}),\end{aligned}$$

the integration is performed over the unit circles (in the counterclockwise direction), and the variables s and t have small moduli.

A similar formula is valid for $P(T_{n,2}; s, t)$. One has just to multiply the above integrand by the function

$$n + \sum_{r=1}^{n-1} \sum_{k=1}^{r} \left(x_k x_{k+1} \cdots x_r + \frac{1}{x_k x_{k+1} \cdots x_r} \right).$$

For $n = 5$ and $n = 6$ we have computed the two types of integrals by using MAPLE [13]. Each of the cases $n = 6$ required about two weeks of computing time on a machine running R10000 CPU at 250 MHz with 8 GB of RAM. □

We have verified independently the low degree (≤ 25) coefficients in the Taylor expansions of $P(C_{5,2}; s, t)$ and $P(C_{6,2}; s, t)$ by using a formula due to Formanek (see Appendix B).

3. Simply graded Poincaré series

Let $P(C_{n,2}; t)$ denote the simply graded Poincaré series of $C_{n,2}$ and $P(T_{n,2}; t)$ the one for $T_{n,2}$. Of course, one has

$$P(C_{n,2}; t) = P(C_{n,2}; t, t), \quad P(T_{n,2}; t) = P(T_{n,2}; t, t). \tag{3.1}$$

By setting $s = t$ in the integral formula for $P(C_{n,2}; s, t)$ resp. $P(T_{n,2}; s, t)$ one obtains a valid formula for $P(C_{n,2}; t)$ resp. $P(T_{n,2}; t)$.

It is much simpler to compute the functions $P(C_{n,2}; t)$ and $P(T_{n,2}; t)$ in the simply graded case than the corresponding functions $P(C_{n,2}; s, t)$ and $P(C_{n,2}; s, t)$ in the bigraded case. Therefore we have first computed the former and then the latter and used the equations (3.1) as a check of correctness.

We can write the above rational functions in lowest terms as

$$P(C_{n,2}; t) = \frac{N(C_{n,2}; t)}{D(C_{n,2}; t)} \tag{3.2}$$

and

$$P(T_{n,2}; t) = \frac{N(T_{n,2}; t)}{D(T_{n,2}; t)}. \tag{3.3}$$

For $n \leq 6$, the above denominators are products of binomials $1 - t^k$ with $k \geq 1$.

Note that $N(C_{n,2}; t)$ and $N(C_{n,2}; t, t)$ may be different because the numerator $N(C_{n,2}; t, t)$ and the denominator $D(C_{n,2}; t, t)$ may have a common factor. Indeed, for $n = 4, 5, 6$ the gcd of these two polynomials is:

$$\begin{aligned} n = 4: &\quad 1 + t + t^2, \\ n = 5: &\quad (1 - t)^2 (1 + t^2)^2, \\ n = 6: &\quad (1 + t)^2 (1 + t + t^2)(1 - t^3)(1 - t^4)^2 (1 - t^5)^3. \end{aligned}$$

Similarly, $N(T_{n,2}; t, t)$ and $D(T_{n,2}; t, t)$ may have a common factor. It turns out that their gcd's are the same as the ones listed above.

In Tables 3 and 4 we list the lowest degree numerators and denominators for the simply graded Poincaré series $P(C_{n,2}; t)$ and $P(T_{n,2}; t)$ for $n \leq 6$. For the

actual power series expansions of the rational functions $P(C_{n,2};t)$ and $P(T_{n,2};t)$ see Appendices C and D (Tables 9 and 11), respectively.

Table 3: Numerators $N(C_{n,2};t)$ and denominators $D(C_{n,2};t)$

$$\begin{aligned}
N(C_{1,2};t) &= N(C_{2,2};t)=1,\\
D(C_{1,2};t) &= (1-t)^2,\\
D(C_{2,2};t) &= (1-t)^2\left(1-t^2\right)^3,\\
N(C_{3,2};t) &= 1-t^2+t^4,\\
D(C_{3,2};t) &= (1-t)^2\left(1-t^2\right)^4\left(1-t^3\right)^4,\\
N(C_{4,2};t) &= \left(1-t^2+t^4\right)\left(1-t-t^3+t^4+2\,t^5+t^6-t^7-t^9+t^{10}\right),\\
D(C_{4,2};t) &= (1-t)^3\left(1-t^2\right)^4\left(1-t^3\right)^5\left(1-t^4\right)^5,\\
N(C_{5,2};t) &= 1+2\,t-6\,t^3-9\,t^4+2\,t^5+25\,t^6+38\,t^7+17\,t^8-34\,t^9-68\,t^{10}\\
&\quad -34\,t^{11}+73\,t^{12}+176\,t^{13}+171\,t^{14}+34\,t^{15}-127\,t^{16}-156\,t^{17}\\
&\quad -2\,t^{18}+218\,t^{19}+322\,t^{20}+218\,t^{21}-\cdots+2\,t^{39}+t^{40},\\
D(C_{5,2};t) &= \left(1-t^2\right)^6\left(1-t^3\right)^8\left(1-t^4\right)^6\left(1-t^5\right)^6,\\
N(C_{6,2};t) &= 1-3\,t+3\,t^2-3\,t^3+3\,t^4+4\,t^5-2\,t^6-8\,t^8-8\,t^9+11\,t^{10}+t^{11}\\
&\quad +56\,t^{12}-24\,t^{13}+48\,t^{14}-69\,t^{15}-9\,t^{16}+2\,t^{17}+78\,t^{18}+118\,t^{19}\\
&\quad +223\,t^{20}+23\,t^{21}+158\,t^{22}-182\,t^{23}+221\,t^{24}-42\,t^{25}+600\,t^{26}\\
&\quad +365\,t^{27}+633\,t^{28}+324\,t^{29}+303\,t^{30}-31\,t^{31}+484\,t^{32}+178\,t^{33}\\
&\quad +1055\,t^{34}+518\,t^{35}+1055\,t^{36}+\cdots-3\,t^{69}+t^{70},\\
D(C_{6,2};t) &= (1-t)^5\left(1-t^2\right)^3\left(1-t^3\right)^6\left(1-t^4\right)^9\left(1-t^5\right)^7\left(1-t^6\right)^7.
\end{aligned}$$

Table 4: Numerators $N(T_{n,2};t)$ and denominators $D(T_{n,2};t)$

$$
\begin{aligned}
N(T_{1,2};t) &= N(T_{2,2};t) = N(T_{3,2};t) = 1,\\
D(T_{1,2};t) &= (1-t)^2\,,\\
D(T_{2,2};t) &= (1-t)^4\left(1-t^2\right),\\
D(T_{3,2};t) &= (1-t)^4\left(1-t^2\right)^4\left(1-t^3\right)^2,\\
N(T_{4,2};t) &= 1-t+t^3+t^5-t^7+t^8,\\
D(T_{4,2};t) &= (1-t)^5\left(1-t^2\right)^4\left(1-t^3\right)^5\left(1-t^4\right)^3,\\
N(T_{5,2};t) &= 1+2\,t+t^2-2\,t^3-t^4+8\,t^5+20\,t^6+24\,t^7+18\,t^8+12\,t^9\\
&\quad +20\,t^{10}+44\,t^{11}+76\,t^{12}+94\,t^{13}+85\,t^{14}+58\,t^{15}+44\,t^{16}\\
&\quad +58\,t^{17}+\cdots+2\,t^{31}+t^{32},\\
D(T_{5,2};t) &= (1-t)^2\left(1-t^2\right)^6\left(1-t^3\right)^8\left(1-t^4\right)^6\left(1-t^5\right)^4,\\
N(T_{6,2};t) &= 1-3\,t+4\,t^2-4\,t^3+4\,t^4+3\,t^5-6\,t^6+11\,t^7-12\,t^8+12\,t^9\\
&\quad +12\,t^{10}+t^{11}+55\,t^{12}-22\,t^{13}+82\,t^{14}+77\,t^{16}+119\,t^{17}+84\,t^{18}\\
&\quad +234\,t^{19}+160\,t^{20}+227\,t^{21}+312\,t^{22}+207\,t^{23}+507\,t^{24}+301\,t^{25}\\
&\quad +612\,t^{26}+469\,t^{27}+517\,t^{28}+593\,t^{29}+426\,t^{30}+593\,t^{31}+\cdots\\
&\quad -3\,t^{59}+t^{60},\\
D(T_{6,2};t) &= (1-t)^7\left(1-t^2\right)^3\left(1-t^3\right)^6\left(1-t^4\right)^9\left(1-t^5\right)^7\left(1-t^6\right)^5.
\end{aligned}
$$

It is often desirable to rewrite the formulae (3.2) and (3.3) as

$$P(C_{n,2};t) = \frac{N^*(C_{n,2};t)}{D^*(C_{n,2};t)} \tag{3.4}$$

and

$$P(T_{n,2};t) = \frac{N^*(T_{n,2};t)}{D^*(T_{n,2};t)}, \tag{3.5}$$

where the numerators have nonnegative integral coefficients, and the denominators are still products of binomials $1-t^k$ with $k \geq 1$. Such forms exist but are not unique. The most interesting ones arise from a choice of a homogeneous system of parameters (HSOP) for the algebra $C_{n,2}$ in which case the multiset of the exponents k is the same as the one made up from degrees of the polynomials in the HSOP. We give several such formulae in Tables 5 and 6 when they differ from those in (3.2) and (3.3). Those in Table 5 do arise from an HSOP except possibly the one for $C_{5,2}$.

Table 5: Numerators $N^*(C_{n,2};t)$ and denominators $D^*(C_{n,2};t)$

$$
\begin{aligned}
N^*(C_{3,2};t) &= \left(1+t^2\right) N(C_{3,2};t) = 1+t^6,\\
D^*(C_{3,2};t) &= (1-t)^2 \left(1-t^2\right)^3 \left(1-t^3\right)^4 \left(1-t^4\right),\\
N^*(C_{4,2};t) &= \left(1+t^2\right) \left(1+t+t^2\right) \left(1+t^3\right)^2 N(C_{4,2};t)\\
&= \left(1+t^6\right) \left(1+2\,t^5+t^6+2\,t^7+4\,t^8+4\,t^9+4\,t^{10}+\cdots+t^{18}\right),\\
&= 1+2\,t^5+2\,t^6+2\,t^7+4\,t^8+4\,t^9+4\,t^{10}+4\,t^{11}+2\,t^{12}\\
&\quad +4\,t^{13}+4\,t^{14}+4\,t^{15}+4\,t^{16}+2\,t^{17}+2\,t^{18}+2\,t^{19}+t^{24},\\
D^*(C_{4,2};t) &= (1-t)^2 \left(1-t^2\right)^3 \left(1-t^3\right)^4 \left(1-t^4\right)^6 \left(1-t^6\right)^2,\\
N^*(C_{5,2};t) &= \left(1-t+t^2\right)^2 \left(1+t^3\right)^2 \left(1+t^2+t^4\right) N(C_{5,2};t)\\
&= 1+2\,t^5+2\,t^6+8\,t^7+13\,t^8+16\,t^9+25\,t^{10}+28\,t^{11}+46\,t^{12}\\
&\quad +58\,t^{13}+85\,t^{14}+132\,t^{15}+172\,t^{16}+232\,t^{17}+282\,t^{18}+346\,t^{19}\\
&\quad +404\,t^{20}+444\,t^{21}+518\,t^{22}+570\,t^{23}+633\,t^{24}+684\,t^{25}\\
&\quad +711\,t^{26}+744\,t^{27}+711\,t^{28}+684\,t^{29}+\cdots+2\,t^{49}+t^{54},\\
D^*(C_{5,2};t) &= (1-t)^2 \left(1-t^2\right)^3 \left(1-t^3\right)^4 \left(1-t^4\right)^6 \left(1-t^5\right)^6 \left(1-t^6\right)^5.
\end{aligned}
$$

Table 6: Numerators $N^*(T_{n,2};t)$ and denominators $D^*(T_{n,2};t)$

$$
\begin{aligned}
N^*(T_{4,2};t) &= \left(1+t+t^2\right) N(T_{4,2};t) = 1+t^4+2\,t^5+t^6+t^{10},\\
D^*(T_{4,2};t) &= (1-t)^4 \left(1-t^2\right)^4 \left(1-t^3\right)^6 \left(1-t^4\right)^3,\\
N^*(T_{5,2};t) &= (1+t) \left(1+t^2\right) N(T_{5,2};t)\\
&= 1+3\,t+4\,t^2+2\,t^3+6\,t^5+25\,t^6+51\,t^7+70\,t^8+74\,t^9\\
&\quad +74\,t^{10}+94\,t^{11}+152\,t^{12}+234\,t^{13}+299\,t^{14}+313\,t^{15}\\
&\quad +281\,t^{16}+245\,t^{17}+245\,t^{18}+281\,t^{19}+\cdots+3\,t^{34}+t^{35},\\
D^*(T_{5,2};t) &= (1-t) \left(1-t^2\right)^6 \left(1-t^3\right)^8 \left(1-t^4\right)^7 \left(1-t^5\right)^4.
\end{aligned}
$$

4. Generators of the algebra $C_{4,2}$

Teranishi [20] has constructed an HSOP for the pure trace algebra $C_{4,2}$. Here is his result:

Theorem 4.1. *Let x and y be generic 4×4 matrices. Then the traces of the 17 matrices (which we arrange according to their degrees)*

$$
\begin{gathered}
x,\ y,\\
x^2,\ xy,\ y^2,\\
x^3,\ x^2y,\ xy^2,\ y^3,\\
x^4,\ x^3y,\ x^2y^2,\ xy^3,\ y^4,\ xyxy,\\
(x^2y)^2,\ (y^2x)^2
\end{gathered}
$$

form an HSOP of the algebra $C_{4,2}$.

However, he did not compute a minimal set of generators for $C_{4,2}$.

The group GL_2 acts (via its standard representation) on the 2-dimensional space spanned by the matrices x and y. This action induces an action on the algebra $C_{4,2}$, which was investigated by Drensky and Sadikova in their recent paper [5]. They show that there is a minimal set of generators whose span is a semisimple graded GL_2-submodule of $C_{4,2}$. Moreover they determine the structure of this module. Its Poincaré polynomial is

$$2t + 3t^2 + 4t^3 + 6t^4 + 2t^5 + 4t^6 + 2t^7 + 4t^8 + 4t^9 + t^{10}.$$

Hence, a minimal generating set consists of 32 polynomials. In their paper they do not list explicitly such a generating set.

We have computed a minimal set of generators of $C_{4,2}$ independently of the above mentioned paper:

Theorem 4.2. *The 17 traces mentioned in Theorem 4.1 together with the traces of the following 15 matrices (arranged by their degrees)*

$$\begin{array}{l} x^3y^2,\ y^3x^2, \\ x^2y^2xy,\ y^2x^2yx, \\ x^3y^2xy,\ y^3x^2yx, \\ x^3y^2x^2y,\ y^3x^2y^2x,\ x^3y^3xy,\ y^3x^3yx, \\ x^3yx^2yxy,\ x^2y^2xyx^2y,\ y^2x^2yxy^2x,\ y^3xy^2xyx, \\ x^3y^3x^2y^2 \end{array}$$

form a minimal set of generators of $C_{4,2}$.

Proof. Our proof is computational. We start with the system of parameters from Teranishi's theorem and consider it as the first approximation to the genuine generating set which we want to construct. Then we compare the Poincaré series of $C_{4,2}$ with that of its subalgebra generated by this HSOP. The difference of the former and the latter is a series with nonnegative integer coefficients. We find the first nonzero term, say ct^d. This means that we have to enlarge our incomplete generating set by additional c generators of degree d. We then select c words in x and y of length d, whose traces provide these additional generators.

We repeat this procedure with the enlarged set of generators, and continue repeating it until we reach the space of invariants of degree 10. After adding the single additional generator in degree 10, we are certain that we have indeed found the full set of generators. This is a consequence of the well known fact [4, Part A, Theorem 6.1.6], originally due to Procesi [15], which establishes the connection between the Nagata–Higman theorem and the invariant theory of generic matrices. □

Remark 4.3. The words y^3x^2yx, $y^3x^2y^2x$, $y^2x^2yxy^2x$, y^3xy^2xyx in Theorem 4.2 can be replaced with xyx^2y^3, $xy^2x^2y^3$, $xy^2xyx^2y^2$, $xyxy^2xy^3$, respectively.

Note that the total number of generators listed in the theorem is 32, and that this number as well as the degrees of these generators are in agreement with the result of Drensky and Sadikova mentioned above.

Let us say that a set W of words in two noncommuting indeterminates x, y is a *test set* for $M_n(\mathbf{C})$ if it has the following property: Two matrices $a, b \in M_n(\mathbf{C})$ are unitarily similar iff $\mathrm{tr}\,(w(a, a^*)) = \mathrm{tr}\,(w(b, b^*))$ holds for all $w \in W$. Such a test set W is *minimal* if no proper subset of W is a test set.

As a consequence of the above theorem, we can state the following criterion for unitary similarity of two 4×4 complex matrices.

Theorem 4.4. *The following 20 words form a test set for $M_4(\mathbf{C})$:*

$$x;\ x^2,\ xy;\ x^3,\ x^2y;\ x^4,\ x^3y,\ x^2y^2,\ xyxy;\ x^3y^2;\ (x^2y)^2,\ x^2y^2xy,\ y^2x^2yx;$$
$$x^3y^2xy;\ x^3y^2x^2y,\ x^3y^3xy,\ y^3x^3yx;\ x^3yx^2yxy,\ x^2y^2xyx^2y;\ x^3y^3x^2y^2.$$

Proof. First apply the above remark. Then among the 32 words $w(x,y)$ whose traces generate the algebra $C_{4,2}$, given in Theorem 4.2, there are 12 pairs $\{w_1, w_2\}$ such that, for any 4×4 complex matrix a, $\operatorname{tr}(w_1(a,a^*))$ and $\operatorname{tr}(w_2(a,a^*))$ are complex conjugates. For instance, $\{x,y\}$, $\{x^2,y^2\}$ and $\{x^3y^2, y^3x^2\}$ are such pairs. (For the non-paired words, such as $w(x,y) = xyxy$, the trace of $w(a,a^*)$ is always real.) By dropping one of the words from each of these pairs, we obtain the test set in the theorem. $\square$

5. Generators of the algebra $C_{5,2}$

Let $M_n(0)$ be the subspace of M_n consisting of matrices of trace 0 and let $C_{n,2}(0) = K[M_n(0)^2]^{\mathrm{GL}_n}$ be the corresponding algebra of GL_n-invariant polynomial functions on the direct product $M_n(0)^2 = M_n(0) \times M_n(0)$. Then one has an isomorphism of $\mathbf{Z}^2$-graded algebras

$$C_{n,2} \cong K[u,v] \otimes C_{n,2}(0), \tag{5.1}$$

where $K[u,v]$ is the polynomial algebra in two variables u and v (see e.g. [15, Section 5]). Consequently, the problem of constructing a minimal set of generators of $C_{n,2}$ reduces to the same problem for $C_{n,2}(0)$. In the remainder of this section we shall assume that $x, y \in M_n(0)$. (More precisely, the entries of x and y will be treated, via restriction, as linear functions on the subspace $M_n(0)$ of M_n.)

In view of the above isomorphism, we have the following obvious relation between the Poincaré series of these algebras:

$$P(C_{n,2};t) = \frac{P(C_{n,2}(0);t)}{(1-t)^2}.$$

Theorem 5.1. *The algebra $C_{5,2}(0)$ has a minimal generating set $\mathcal{P}$ consisting of 171 bihomogeneous polynomials. $\mathcal{P}$ is the disjoint union of four subsets: $\mathcal{P}_s$, $\mathcal{P}_k$, $\mathcal{P}_d$ and $\mathcal{P}'_d$ with cardinals 5, 4, 81 and 81, respectively. The polynomials $f(x,y) \in \mathcal{P}_s$ are symmetric, (i.e., satisfy $f(y,x) = f(x,y)$), while those in $\mathcal{P}_k$ are skew-symmetric. The polynomials in $\mathcal{P}_d$ are neither symmetric nor skew-symmetric. There is a bijection $\mathcal{P}_d \to \mathcal{P}'_d$ given by $f(x,y) \to f(y,x)$. The sets $\mathcal{P}_s$, $\mathcal{P}_k$ and $\mathcal{P}_d$ are given below.*

$\mathcal{P}_s$ consists of the traces of the following 5 words in x and y:

$$xy,\ x^2y^2,\ xyxy,\ x^3y^3,\ x^4y^4.$$

$\mathcal{P}_k$ consists of the traces of the following 4 matrices:

$$x^3y^3x^2y^2 - y^3x^3y^2x^2,\ x^2yxy^2xyxy - y^2xyx^2yxyx,$$
$$x^3y^2xyxy^2xy - y^3x^2yxyx^2yx,\ x^4y^4x^3y^3 - y^4x^4y^3x^3.$$

Finally, the set $\mathcal{P}_d$ consists of the traces of the following 81 words:

x^2;
x^3, x^2y;
x^4, x^3y;
x^5, x^4y, x^3y^2, x^2yxy;
x^4y^2, $(x^2y)^2$, x^2y^2xy;
x^4y^3, x^4yxy, x^3y^2xy, x^3yxy^2;
x^4y^2xy, x^4yx^2y, x^4yxy^2, x^3y^3xy, $x^3y^2x^2y$, x^2y^2xyxy;
$x^4y^2x^2y$, $x^4y^2xy^2$, $x^4yx^2y^2$, $x^3y^3x^2y$, x^3y^2xyxy, x^3yxyx^2y, x^3yxyxy^2, $x^2y^2xyx^2y$;
x^4y^4xy, $x^4y^3xy^2$, $x^4y^2x^3y$, $x^4y^2x^2y^2$, x^4y^2xyxy, x^4yx^2yxy, x^4yxy^2xy, x^3y^3xyxy, $x^3y^2x^2yxy$, $x^3yx^2yxy^2$;
$x^4y^4x^2y$, $x^4y^4xy^2$, $x^4y^3x^3y$, $x^4y^3x^2y^2$, x^4y^3xyxy, $x^4y^2x^2yxy$, $x^4y^2xyxy^2$, x^4yx^3yxy, $x^4yx^2y^2xy$, $x^4yx^2yxy^2$, $x^3y^2x^2y^2xy$, x^3yxyxy^2xy;
$x^4y^4x^3y$, $x^4y^4x^2y^2$, $x^4y^3x^3y^2$, $x^4y^3x^2yxy$, $x^4y^3xy^2xy$, $x^4y^2x^3yxy$, $x^4y^2x^2yx^2y$, $x^4y^2xy^2x^2y$, $x^4y^2xyx^2y^2$, $x^4yx^3yx^2y$, $x^4yx^3yxy^2$, x^4yxyxy^2xy, $x^3y^3x^2yxy^2$;
$x^4y^4x^3y^2$, $x^4y^4x^2yxy$, $x^4y^4xy^2xy$, $x^4y^3x^3yxy$, $x^4y^3x^2y^2xy$, $x^4y^3(x^2y)^2$, $x^4y^2x^3yx^2y$, $x^4(y^2x^2)^2y$, $x^4y^2xyxy^2xy$;
$x^4y^4x^3yxy$, $x^4y^4x^2y^2xy$, $x^4y^3x^3y^2xy$, $x^4y^3x^2yx^3y$;
$x^4y^4x^3y^2xy$, $x^4y^3x^3y^2x^2y$, $x^4y^2x^3yx^2yxy$.

Proof. Shestakov and Zhukavets [17] have shown that any 2-generated associative algebra (non-unital and over a field of characteristic 0) which satisfies the identity $x^5 = 0$, also satisfies the identity $x_1x_2\cdots x_{15} = 0$. Since we are working with only two generic matrices, by invoking a theorem of Procesi [15, Theorem 3.2], we conclude that the algebra $C_{5,2}$ (and $C_{5,2}(0)$) is generated by polynomials of degree at most 15.

Let $\mathcal{A}$ denote the unital subalgebra of $C_{5,2}(0)$ generated by 171 polynomials in our set $\mathcal{P}$. We have verified, using a computer, that for each degree $d \leq 15$ the homogeneous component, $\mathcal{A}_d$, of $\mathcal{A}$ of degree d has the dimension equal to the coefficient of t^d in the Poincaré series of the algebra $C_{5,2}(0)$. For the reader's convenience, these coefficients are given in Appendix C. As an additional check, we have computed the dimension of $\mathcal{A}_{16}$ and verified that it is indeed equal to 17338. $\square$

Remark 5.2. In view of (5.1), it follows that a minimal generating set of the algebra $C_{5,2}$ has cardinal 173.

Remark 5.3. One can modify the minimal generating set $\mathcal{P}$ by replacing each of the four generators $\mathrm{tr}\ (w(x,y)-w(y,x)) \in \mathcal{P}_k$ with $\mathrm{tr}\ (w(x,y))$. The only reason for our choice was to make $\mathcal{P}$ stable (up to sign) under the involution which interchanges x and y.

Remark 5.4. By using this theorem, one can now construct easily a test set for $M_5(\mathbf{C})$ of cardinal $5 + 4 + 82 = 91$.

6. Poincaré series for invariants of $\mathrm{GL}_n \times \mathrm{SL}_d$

The vector space M_n^d can be identified with the tensor product $M_n \otimes K^d$ of M_n and the d-dimensional space K^d. The action of GL_n on M_n^d corresponds to its action on this tensor product given by $a \cdot (x \otimes v) = axa^{-1} \otimes v$. We can now view $M_n \otimes K^d$ as a module for the direct product $\mathrm{GL}_n \times \mathrm{GL}_d$ by letting GL_d act on K^d by multiplication. Denote by $C_{n,d}^{\#}$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n \times \mathrm{SL}_d$-invariant functions. This is a subalgebra of $C_{n,d}$.

Our objective in this section is to record the Poincaré series of the algebras $C_{n,2}^{\#}$ for $n \leq 5$. We shall write these Poincaré series as rational functions in lowest terms

$$P(C_{n,2}^{\#};t) = \frac{N(C_{n,2}^{\#};t)}{D(C_{n,2}^{\#};t)},$$

with numerator and denominator normalized so to take value 1 at $t = 0$. These numerators are again palindromic.

Table 7: Numerators $N(C_{n,2}^{\#};t)$ and denominators $D(C_{n,2}^{\#};t)$

$$\begin{aligned}
N(C_{2,2}^{\#};t) &= 1,\\
D(C_{2,2}^{\#};t) &= \left(1-t^4\right)^2,\\
N(C_{3,2}^{\#};t) &= 1+3\,t^8+t^{10}+3\,t^{12}+t^{20},\\
D(C_{3,2}^{\#};t) &= \left(1-t^4\right)^3\left(1-t^6\right)^3\left(1-t^8\right),\\
N(C_{4,2}^{\#};t) &= 1+10\,t^8+12\,t^{10}+38\,t^{12}+46\,t^{14}+93\,t^{16}+131\,t^{18}+235\,t^{20}\\
&\quad +299\,t^{22}+473\,t^{24}+560\,t^{26}+714\,t^{28}+761\,t^{30}+876\,t^{32}\\
&\quad +830\,t^{34}+876\,t^{36}+\cdots+12\,t^{58}+10\,t^{60}+t^{68},\\
D(C_{4,2}^{\#};t) &= \left(1-t^4\right)^3\left(1-t^6\right)^4\left(1-t^8\right)^4\left(1-t^{10}\right)^2\left(1-t^{12}\right),\\
N(C_{5,2}^{\#};t) &= 1+t^2+t^4+t^6+14\,t^8+41\,t^{10}+135\,t^{12}+329\,t^{14}+842\,t^{16}\\
&\quad +1980\,t^{18}+4677\,t^{20}+10386\,t^{22}+22654\,t^{24}+47093\,t^{26}\\
&\quad +94970\,t^{28}+184182\,t^{30}+346523\,t^{32}+629769\,t^{34}+1111589\,t^{36}\\
&\quad +1902191\,t^{38}+3165521\,t^{40}+5120359\,t^{42}+8066607\,t^{44}\\
&\quad +12376177\,t^{46}+18520117\,t^{48}+27035364\,t^{50}+38541637\,t^{52}\\
&\quad +53673328\,t^{54}+73078953\,t^{56}+97307914\,t^{58}+126802726\,t^{60}\\
&\quad +161749890\,t^{62}+202084191\,t^{64}+247338162\,t^{66}+296695937\,t^{68}\\
&\quad +348874713\,t^{70}+402270954\,t^{72}+454898759\,t^{74}+504632564\,t^{76}\\
&\quad +549206297\,t^{78}+586521387\,t^{80}+614654835\,t^{82}+632178319\,t^{84}\\
&\quad +638112785\,t^{86}+632178319\,t^{88}+\cdots+14\,t^{164}+t^{166}+t^{168}\\
&\quad +t^{170}+t^{172},\\
D(C_{5,2}^{\#};t) &= \left(1+t^2+t^4\right)\left(1-t^4\right)^3\left(1-t^6\right)^3\left(1-t^8\right)^5\left(1-t^{10}\right)^5\left(1-t^{12}\right)^3\cdot\\
&\quad \left(1-t^{14}\right)^2\left(1-t^{16}\right)\left(1-t^{18}\right).
\end{aligned}$$

These rational functions were again computed by using the Molien–Weyl formula. In this case the formula is more complicated:

$$P(C^{\#}_{n,2};t) = \frac{1}{2\pi i}\int_{|y|=1} \frac{(1-y^2)\psi(y)}{(1-ty)^n(1-\frac{t}{y})^n}\frac{\mathrm{d}y}{y},$$

where

$$\psi(y) = \frac{1}{(2\pi i)^{n-1}}\int_{|x_1|=1}\cdots\int_{|x_{n-1}|=1}\prod_{1\le k\le r\le n-1}\frac{1-x_kx_{k+1}\cdots x_r}{\psi_{k,r}}\frac{\mathrm{d}x_{n-1}}{x_{n-1}}\cdots\frac{\mathrm{d}x_1}{x_1},$$

$$\begin{aligned}\psi_{k,r} \quad = \quad & (1-tyx_kx_{k+1}\cdots x_r)(1-ty^{-1}x_kx_{k+1}\cdots x_r)\cdot \\ & (1-ty(x_kx_{k+1}\cdots x_r)^{-1})(1-t(yx_kx_{k+1}\cdots x_r)^{-1})\end{aligned}$$

and the integration is performed over the unit circles assuming that $|t|<1$.

The computations were easy for $n\le 4$ but very hard (lasting several days) for $n=5$. We close this section by listing the terms of degree <26 in the Taylor series of these rational functions.

$$\begin{aligned}
P(C^{\#}_{2,2};t) &= 1+2\,t^4+3\,t^8+4\,t^{12}+5\,t^{16}+6\,t^{20}+7\,t^{24}+\cdots, \\
P(C^{\#}_{3,2};t) &= 1+3\,t^4+3\,t^6+10\,t^8+10\,t^{10}+31\,t^{12}+33\,t^{14}+73\,t^{16}+92\,t^{18} \\
&\quad +164\,t^{20}+205\,t^{22}+344\,t^{24}+\cdots, \\
P(C^{\#}_{4,2};t) &= 1+3\,t^4+4\,t^6+20\,t^8+26\,t^{10}+101\,t^{12}+168\,t^{14}+445\,t^{16} \\
&\quad +813\,t^{18}+1804\,t^{20}+3246\,t^{22}+6527\,t^{24}+\cdots, \\
P(C^{\#}_{5,2};t) &= 1+3\,t^4+4\,t^6+24\,t^8+44\,t^{10}+171\,t^{12}+388\,t^{14}+1166\,t^{16} \\
&\quad +2808\,t^{18}+7344\,t^{20}+17281\,t^{22}+41569\,t^{24}+\cdots.
\end{aligned}$$

Since the algebra $C^{\#}_{5,2}$ is the algebra of SL_2-invariants in $C_{5,2}$, the coefficients of t^{2k} in the above Taylor series must be the same as the coefficients of the Schur functions $f_{k,k}$ in the formula for $P(C_{5,2};t)$ displayed in Appendix B. It is easy to check that this is indeed the case for $k\le 12$, which gives a further confirmation of our formula for $P(C^{\#}_{5,2};t)$.

7. Poincaré series for invariants of $\mathrm{GL}_n\times\Delta_{d-1}$

Let us restrict the action of $\mathrm{GL}_n\times\mathrm{SL}_d$ on M_n^d to its subgroup $\mathrm{GL}_n\times\Delta_{d-1}$, where Δ_{d-1} is the maximal torus of SL_d consisting of diagonal matrices. Denote by $C^{\bullet}_{n,d}$ the subalgebra of $K[M_n^d]$ consisting of $\mathrm{GL}_n\times\Delta_{n-1}$-invariant polynomial functions. This is a subalgebra of $C_{n,d}$.

In this section we record the Poincaré series of $C^{\bullet}_{n,2}$ for $n\le 5$. We shall write these rational functions in lowest terms as

$$P(C^{\bullet}_{n,2};t) = \frac{N(C^{\bullet}_{n,2};t)}{D(C^{\bullet}_{n,2};t)},$$

with numerator and denominator normalized so to take value 1 at $t=0$. These numerators are again palindromic.

Table 8: Numerators $N(C^\bullet_{n,2};t)$ and denominators $D(C^\bullet_{n,2};t)$

$$N(C^\bullet_{1,2};t) = 1,$$
$$D(C^\bullet_{1,2};t) = 1-t^2,$$
$$N(C^\bullet_{2,2};t) = 1+t^4,$$
$$D(C^\bullet_{2,2};t) = \left(1-t^2\right)^2\left(1-t^4\right)^2,$$
$$N(C^\bullet_{3,2};t) = 1+3\,t^4+6\,t^6+9\,t^8+6\,t^{10}+12\,t^{12}+6\,t^{14}+\cdots+t^{24},$$
$$D(C^\bullet_{3,2};t) = \left(1-t^2\right)^2\left(1-t^4\right)^3\left(1-t^6\right)^3\left(1-t^8\right),$$
$$N(C^\bullet_{4,2};t) =$$
$$1+4\,t^4+12\,t^6+36\,t^8+68\,t^{10}+171\,t^{12}+316\,t^{14}+639\,t^{16}+1096\,t^{18}$$
$$+1096\,t^{18}+1849\,t^{20}+2794\,t^{22}+4151\,t^{24}+5546\,t^{26}+7229\,t^{28}+8700\,t^{30}$$
$$+10085\,t^{32}+10836\,t^{34}+11270\,t^{36}+10836\,t^{38}+\cdots+12\,t^{66}+4\,t^{68}+t^{72},$$
$$D(C^\bullet_{4,2};t) = \left(1-t^2\right)^2\left(1-t^4\right)^3\left(1-t^6\right)^4\left(1-t^8\right)^4\left(1-t^{10}\right)^2\left(1-t^{12}\right),$$
$$N(C^\bullet_{5,2};t) =$$
$$1+t^2+5\,t^4+20\,t^6+76\,t^8+227\,t^{10}+692\,t^{12}+1933\,t^{14}+5307\,t^{16}+13752\,t^{18}$$
$$+34304\,t^{20}+81525\,t^{22}+186346\,t^{24}+408071\,t^{26}+860437\,t^{28}+1746504\,t^{30}$$
$$+3421732\,t^{32}+6474866\,t^{34}+11857662\,t^{36}+21033945\,t^{38}+36195856\,t^{40}$$
$$+60479854\,t^{42}+98242554\,t^{44}+155273212\,t^{46}+239019423\,t^{48}+358621723\,t^{50}$$
$$+524884888\,t^{52}+749897456\,t^{54}+1046516425\,t^{56}+1427383948\,t^{58}$$
$$+1903851664\,t^{60}+2484438301\,t^{62}+3173436196\,t^{64}+3969248353\,t^{66}$$
$$+4863282209\,t^{68}+5838905156\,t^{70}+6871421892\,t^{72}+7928353846\,t^{74}$$
$$+8971036674\,t^{76}+9956478001\,t^{78}+10840418189\,t^{80}+11580232480\,t^{82}$$
$$+12138549745\,t^{84}+12485984964\,t^{86}+12603960344\,t^{88}+12485984964\,t^{90}$$
$$+\cdots+20\,t^{170}+5\,t^{172}+t^{174}+t^{176},$$
$$D(C^\bullet_{5,2};t) =$$
$$\left(1-t^2\right)\left(1-t^4\right)^3\left(1-t^6\right)^4\left(1-t^8\right)^5\left(1-t^{10}\right)^5\left(1-t^{12}\right)^3\left(1-t^{14}\right)^2\left(1-t^{16}\right)\cdot$$
$$\left(1-t^{18}\right).$$

These rational functions were computed by using the formula:

$$P(C^\bullet_{n,2};t) = \frac{1}{2\pi i}\int_{|z|=1} P(C_{n,2};tz^{-1},tz)\frac{\mathrm{d}z}{z},$$

where $P(C_{n,2};s,t)$ is the bigraded Poincaré series of $C_{n,2}$ from Section 2. We list the terms of degree < 26 in their Taylor expansions:

$$
\begin{aligned}
P(C^{\bullet}_{1,2};t) &= 1+t^2+t^4+t^6+t^8+t^{10}+t^{12}+t^{14}+t^{16}+t^{18}+t^{20}+t^{22}\\
&\quad +t^{24}+\cdots,\\
P(C^{\bullet}_{2,2};t) &= 1+2\,t^2+6\,t^4+10\,t^6+19\,t^8+28\,t^{10}+44\,t^{12}+60\,t^{14}+85\,t^{16}\\
&\quad +110\,t^{18}+146\,t^{20}+182\,t^{22}+231\,t^{24}+\cdots,\\
P(C^{\bullet}_{3,2};t) &= 1+2\,t^2+9\,t^4+25\,t^6+66\,t^8+149\,t^{10}+329\,t^{12}+650\,t^{14}\\
&\quad +1248\,t^{16}+2255\,t^{18}+3941\,t^{20}+6608\,t^{22}+10799\,t^{24}+\cdots,\\
P(C^{\bullet}_{4,2};t) &= 1+2\,t^2+10\,t^4+34\,t^6+116\,t^8+332\,t^{10}+948\,t^{12}+2450\,t^{14}\\
&\quad +6126\,t^{16}+14426\,t^{18}+32746\,t^{20}+71100\,t^{22}+149402\,t^{24}+\cdots,\\
P(C^{\bullet}_{5,2};t) &= 1+2\,t^2+10\,t^4+37\,t^6+143\,t^8+478\,t^{10}+1588\,t^{12}+4910\,t^{14}\\
&\quad +14748\,t^{16}+42235\,t^{18}+116910\,t^{20}+311478\,t^{22}+803343\,t^{24}\\
&\quad +\cdots.
\end{aligned}
$$

8. Conjectures

On the basis of our computations, we propose four conjectures about the numerators $N(C_{n,2};s,t)$ and $N(T_{n,2};s,t)$ and the denominators $D(C_{n,2};s,t)$ and $D(T_{n,2};s,t)$. See Section 2 for their definitions.

Conjecture 8.1. *The denominators $D(C_{n,2};s,t)$ and $D(T_{n,2};s,t)$ can be written as products of binomials $1-s^at^b$, where a and b are nonnegative integers.*

Conjecture 8.2. $N(C_{n,2};1,1)=N(T_{n,2};1,1)=0$ *for $n\geq 5$.*

Conjecture 8.3. *For all n,*

$$(1-s)(1-t)D(C_{n,2};s,t)=(1-s^n)(1-t^n)D(T_{n,2};s,t).$$

Conjecture 8.4. *For all n,*

$$\gcd\left(N(C_{n,2};t,t),D(C_{n,2};t,t)\right)=\gcd\left(N(T_{n,2};t,t),D(T_{n,2};t,t)\right).$$

All four conjectures are true for $n\leq 6$. The second conjecture implies (see the proof of Proposition 2.1) that $C_{n,2}$ has no bigraded system of parameters for $n\geq 5$. As all coefficients of the numerators $N(C^{\#}_{n,2};t)$ and $N(C^{\bullet}_{n,2};t)$ given in Sections 6 and 7 are nonnegative, we propose yet another conjecture.

Conjecture 8.5. *For all n, $N(C^{\#}_{n,2};t)$ and $N(C^{\bullet}_{n,2};t)$ have nonnegative integer coefficients.*

The following interesting problem may have some practical applications.

Problem 8.6. Construct minimal test sets for $M_4(\mathbf{C})$ and $M_5(\mathbf{C})$.

9. Appendix A: Bigraded Poincaré series for $C_{6,2}$ and $T_{6,2}$

The numerator $N(C_{6,2};s,t)$ is a polynomial in s and t of total degree 100, with leading term $(st)^{50}$, having in total 1169 terms. By using its symmetry properties, we can write it as follows

$$N(C_{6,2};s,t)=f(s,t)+(st)^{50}f(s^{-1},t^{-1})-18142(st)^{25},$$

where

$$
\begin{aligned}
f(s,t) &= f_1(st) - 2s^{17}t^{33} f_2(st^{-1}) - st^2 f_3(s,t) - s^2 t f_3(t,s),\\
f_1(x) &= 1 - 2\,x + 9\,x^3 - 3\,x^4 - 35\,x^5 + 27\,x^6 + 137\,x^7 - 89\,x^8 - 388\,x^9\\
&\quad +337\,x^{10} + 955\,x^{11} - 704\,x^{12} - 2155\,x^{13} + 1319\,x^{14} + 4002\,x^{15}\\
&\quad -2209\,x^{16} - 6963\,x^{17} + 2820\,x^{18} + 10659\,x^{19} - 2638\,x^{20} - 14080\,x^{21}\\
&\quad +2275\,x^{22} + 16918\,x^{23} - 1114\,x^{24},\\
f_2(x) &= 10 + 77\,x + 345\,x^2 + 1073\,x^3 + 2480\,x^4 + 4519\,x^5 + 6700\,x^6 + 8497\,x^7,
\end{aligned}
$$

and the polynomial f_3 has 276 terms:

$$
\begin{aligned}
&f_3(s,t) =\\
&2 + 2\,t + t^2 - 4\,st - 7\,st^2 - 8\,st^3 - 8\,s^2t^2 - 5\,st^4 - s^2t^3 - 3\,st^5 + 10\,s^2t^4\\
&+21\,s^3t^3 - st^6 + 15\,s^2t^5 + 32\,s^3t^4 + 15\,s^2t^6 + 25\,s^3t^5 + 25\,s^4t^4 + 10\,s^2t^7\\
&+4\,s^3t^6 - 18\,s^4t^5 + 5\,s^2t^8 - 13\,s^3t^7 - 61\,s^4t^6 - 94\,s^5t^5 + 2\,s^2t^9 - 24\,s^3t^8\\
&-75\,s^4t^7 - 121\,s^5t^6 - 21\,s^3t^9 - 61\,s^4t^8 - 74\,s^5t^7 - 80\,s^6t^6 - 15\,s^3t^{10} - 23\,s^4t^9\\
&+10\,s^5t^8 + 73\,s^6t^7 - 6\,s^3t^{11} + 6\,s^4t^{10} + 90\,s^5t^9 + 201\,s^6t^8 + 290\,s^7t^7 - 2\,s^3t^{12}\\
&+23\,s^4t^{11} + 111\,s^5t^{10} + 241\,s^6t^9 + 326\,s^7t^8 + 21\,s^4t^{12} + 91\,s^5t^{11} + 152\,s^6t^{10}\\
&+173\,s^7t^9 + 170\,s^8t^8 + 13\,s^4t^{13} + 43\,s^5t^{12} + 22\,s^6t^{11} - 111\,s^7t^{10} - 233\,s^8t^9\\
&+5\,s^4t^{14} + 2\,s^5t^{13} - 90\,s^6t^{12} - 318\,s^7t^{11} - 595\,s^8t^{10} - 808\,s^9t^9 + s^4t^{15}\\
&-14\,s^5t^{14} - 125\,s^6t^{13} - 367\,s^7t^{12} - 629\,s^8t^{11} - 902\,s^9t^{10} - 14\,s^5t^{15} - 94\,s^6t^{14}\\
&-248\,s^7t^{13} - 357\,s^8t^{12} - 430\,s^9t^{11} - 420\,s^{10}t^{10} - 7\,s^5t^{16} - 45\,s^6t^{15} - 70\,s^7t^{14}\\
&+48\,s^8t^{13} + 276\,s^9t^{12} + 608\,s^{10}t^{11} - 2\,s^5t^{17} - 7\,s^6t^{16} + 56\,s^7t^{15} + 347\,s^8t^{14}\\
&+812\,s^9t^{13} + 1416\,s^{10}t^{12} + 1807\,s^{11}t^{11} + 7\,s^6t^{17} + 95\,s^7t^{16} + 401\,s^8t^{15}\\
&+887\,s^9t^{14} + 1500\,s^{10}t^{13} + 1947\,s^{11}t^{12} + 6\,s^6t^{18} + 70\,s^7t^{17} + 274\,s^8t^{16}\\
&+535\,s^9t^{15} + 830\,s^{10}t^{14} + 943\,s^{11}t^{13} + 984\,s^{12}t^{12} + 2\,s^6t^{19} + 29\,s^7t^{18}\\
&+101\,s^8t^{17} + 69\,s^9t^{16} - 131\,s^{10}t^{15} - 593\,s^{11}t^{14} - 1082\,s^{12}t^{13} + 5\,s^7t^{19}\\
&-15\,s^8t^{18} - 238\,s^9t^{17} - 780\,s^{10}t^{16} - 1717\,s^{11}t^{15} - 2716\,s^{12}t^{14} - 3416\,s^{13}t^{13}\\
&-2\,s^7t^{20} - 44\,s^8t^{19} - 299\,s^9t^{18} - 857\,s^{10}t^{17} - 1803\,s^{11}t^{16} - 2859\,s^{12}t^{15}\\
&-3687\,s^{13}t^{14} - s^7t^{21} - 30\,s^8t^{20} - 193\,s^9t^{19} - 529\,s^{10}t^{18} - 1016\,s^{11}t^{17}\\
&-1488\,s^{12}t^{16} - 1789\,s^{13}t^{15} - 1873\,s^{14}t^{14} - 9\,s^8t^{21} - 69\,s^9t^{20} - 110\,s^{10}t^{19}\\
&-31\,s^{11}t^{18} + 396\,s^{12}t^{17} + 1118\,s^{13}t^{16} + 1866\,s^{14}t^{15} - s^8t^{22} - s^9t^{21} + 122\,s^{10}t^{20}\\
&+582\,s^{11}t^{19} + 1589\,s^{12}t^{18} + 3168\,s^{13}t^{17} + 4811\,s^{14}t^{16} + 5888\,s^{15}t^{15} + 12\,s^9t^{22}\\
&+155\,s^{10}t^{21} + 630\,s^{11}t^{20} + 1669\,s^{12}t^{19} + 3250\,s^{13}t^{18} + 5064\,s^{14}t^{17} + 6457\,s^{15}t^{16}\\
&+5\,s^9t^{23} + 87\,s^{10}t^{22} + 362\,s^{11}t^{21} + 928\,s^{12}t^{20} + 1783\,s^{13}t^{19} + 2630\,s^{14}t^{18}
\end{aligned}
$$

$$
\begin{aligned}
&+3315\,s^{15}t^{17}+3597\,s^{16}t^{16}+s^{9}t^{24}+25\,s^{10}t^{23}+85\,s^{11}t^{22}+113\,s^{12}t^{21}\\
&-44\,s^{13}t^{20}-589\,s^{14}t^{19}-1536\,s^{15}t^{18}-2424\,s^{16}t^{17}+3\,s^{10}t^{24}-43\,s^{11}t^{23}\\
&-310\,s^{12}t^{22}-1085\,s^{13}t^{21}-2662\,s^{14}t^{20}-4897\,s^{15}t^{19}-7265\,s^{16}t^{18}\\
&-8700\,s^{17}t^{17}-s^{10}t^{25}-47\,s^{11}t^{24}-323\,s^{12}t^{23}-1104\,s^{13}t^{22}-2730\,s^{14}t^{21}\\
&-5139\,s^{15}t^{20}-7802\,s^{16}t^{19}-9930\,s^{17}t^{18}-22\,s^{11}t^{25}-158\,s^{12}t^{24}\\
&-581\,s^{13}t^{23}-1465\,s^{14}t^{22}-2797\,s^{15}t^{21}-4285\,s^{16}t^{20}-5471\,s^{17}t^{19}\\
&-6163\,s^{18}t^{18}-5\,s^{11}t^{26}-39\,s^{12}t^{25}-85\,s^{13}t^{24}-135\,s^{14}t^{23}+38\,s^{15}t^{22}\\
&+557\,s^{16}t^{21}+1452\,s^{17}t^{20}+2361\,s^{18}t^{19}-s^{11}t^{27}+5\,s^{12}t^{26}+103\,s^{13}t^{25}\\
&+519\,s^{14}t^{24}+1588\,s^{15}t^{23}+3626\,s^{16}t^{22}+6408\,s^{17}t^{21}+9221\,s^{18}t^{20}\\
&+10970\,s^{19}t^{19}+6\,s^{12}t^{27}+97\,s^{13}t^{26}+476\,s^{14}t^{25}+1579\,s^{15}t^{24}+3698\,s^{16}t^{23}\\
&+6873\,s^{17}t^{22}+10313\,s^{18}t^{21}+12991\,s^{19}t^{20}+2\,s^{12}t^{28}+38\,s^{13}t^{27}+213\,s^{14}t^{26}\\
&+745\,s^{15}t^{25}+1914\,s^{16}t^{24}+3672\,s^{17}t^{23}+5792\,s^{18}t^{22}+7543\,s^{19}t^{21}\\
&+8636\,s^{20}t^{20}+8\,s^{13}t^{28}+26\,s^{14}t^{27}+64\,s^{15}t^{26}+49\,s^{16}t^{25}-82\,s^{17}t^{24}\\
&-581\,s^{18}t^{23}-1285\,s^{19}t^{22}-1988\,s^{20}t^{21}-22\,s^{14}t^{28}-179\,s^{15}t^{27}-735\,s^{16}t^{26}\\
&-2110\,s^{17}t^{25}-4476\,s^{18}t^{24}-7718\,s^{19}t^{23}-10727\,s^{20}t^{22}-12706\,s^{21}t^{21}\\
&-16\,s^{14}t^{29}-133\,s^{15}t^{28}-624\,s^{16}t^{27}-1950\,s^{17}t^{26}-4569\,s^{18}t^{25}-8302\,s^{19}t^{24}\\
&-12459\,s^{20}t^{23}-15619\,s^{21}t^{22}-4\,s^{14}t^{30}-45\,s^{15}t^{29}-239\,s^{16}t^{28}-862\,s^{17}t^{27}\\
&-2228\,s^{18}t^{26}-4481\,s^{19}t^{25}-7120\,s^{20}t^{24}-9652\,s^{21}t^{23}-11078\,s^{22}t^{22}\\
&-3\,s^{15}t^{30}-9\,s^{16}t^{29}-10\,s^{17}t^{28}+4\,s^{18}t^{27}+100\,s^{19}t^{26}+291\,s^{20}t^{25}\\
&+665\,s^{21}t^{24}+850\,s^{22}t^{23}+3\,s^{15}t^{31}+41\,s^{16}t^{30}+238\,s^{17}t^{29}+892\,s^{18}t^{28}\\
&+2388\,s^{19}t^{27}+4918\,s^{20}t^{26}+8085\,s^{21}t^{25}+11130\,s^{22}t^{24}+12956\,s^{23}t^{23}.
\end{aligned}
$$

The numerator $N(T_{6,2};s,t)$ is a polynomial in s and t of total degree 90, with leading term $(st)^{45}$, having in total 854 terms. By using its symmetry properties, we can write it as follows

$$
\begin{aligned}
N(T_{6,2};s,t) &= (1+st)\left[g_1(st)+(st)^{44}g_1(s^{-1}t^{-1})\right]\\
&\quad -st^2\left[g_2(s,t)+(st)^{42}g_2(t^{-1},s^{-1})\right]-s^2t\left[g_2(t,s)+(st)^{42}g_2(s^{-1},t^{-1})\right],
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(x) \quad = \quad & 1-2\,x+2\,x^2+4\,x^3-3\,x^4-6\,x^5+33\,x^6+32\,x^7-59\,x^8-22\,x^9\\
&+166\,x^{10}-8\,x^{11}-360\,x^{12}-110\,x^{13}+494\,x^{14}+28\,x^{15}-711\,x^{16}\\
&+192\,x^{17}+1203\,x^{18}+80\,x^{19}-1282\,x^{20}-188\,x^{21}+528\,x^{22}
\end{aligned}
$$

and the polynomial g_2 has 206 terms:

$$
\begin{aligned}
&g_2(s,t) = \\
&1 + t - 3\,st - 4\,st^2 - 4\,st^3 - 6\,s^2t^2 - 2\,st^4 - 2\,s^2t^3 - st^5 + 3\,s^2t^4 + 7\,s^3t^3 \\
&+4\,s^2t^5 + 9\,s^3t^4 + 4\,s^2t^6 + 5\,s^3t^5 + s^4t^4 + 2\,s^2t^7 - 3\,s^3t^6 - 19\,s^4t^5 + s^2t^8 \\
&-6\,s^3t^7 - 30\,s^4t^6 - 52\,s^5t^5 - 7\,s^3t^8 - 28\,s^4t^7 - 53\,s^5t^6 - 4\,s^3t^9 - 17\,s^4t^8 \\
&-27\,s^5t^7 - 35\,s^6t^6 - 2\,s^3t^{10} - 3\,s^4t^9 + 6\,s^5t^8 + 22\,s^6t^7 + 4\,s^4t^{10} + 27\,s^5t^9 \\
&+56\,s^6t^8 + 78\,s^7t^7 + 6\,s^4t^{11} + 29\,s^5t^{10} + 60\,s^6t^9 + 73\,s^7t^8 + 3\,s^4t^{12} + 19\,s^5t^{11} \\
&+31\,s^6t^{10} + 19\,s^7t^9 - 5\,s^8t^8 + s^4t^{13} + 7\,s^5t^{12} + s^6t^{11} - 44\,s^7t^{10} - 104\,s^8t^9 \\
&-s^5t^{13} - 18\,s^6t^{12} - 71\,s^7t^{11} - 149\,s^8t^{10} - 213\,s^9t^9 - s^5t^{14} - 17\,s^6t^{13} - 56\,s^7t^{12} \\
&-102\,s^8t^{11} - 148\,s^9t^{10} - s^5t^{15} - 8\,s^6t^{14} - 20\,s^7t^{13} - 5\,s^8t^{12} + 29\,s^9t^{11} \\
&+67\,s^{10}t^{10} - 2\,s^6t^{15} + 10\,s^7t^{14} + 73\,s^8t^{13} + 191\,s^9t^{12} + 318\,s^{10}t^{11} + s^6t^{16} \\
&+18\,s^7t^{15} + 100\,s^8t^{14} + 246\,s^9t^{13} + 428\,s^{10}t^{12} + 543\,s^{11}t^{11} + 12\,s^7t^{16} + 69\,s^8t^{15} \\
&+185\,s^9t^{14} + 324\,s^{10}t^{13} + 434\,s^{11}t^{12} + 4\,s^7t^{17} + 29\,s^8t^{16} + 68\,s^9t^{15} + 111\,s^{10}t^{14} \\
&+109\,s^{11}t^{13} + 96\,s^{12}t^{12} - 20\,s^9t^{16} - 86\,s^{10}t^{15} - 198\,s^{11}t^{14} - 335\,s^{12}t^{13} \\
&-6\,s^8t^{18} - 49\,s^9t^{17} - 155\,s^{10}t^{16} - 331\,s^{11}t^{15} - 519\,s^{12}t^{14} - 653\,s^{13}t^{13} - 3\,s^8t^{19} \\
&-34\,s^9t^{18} - 118\,s^{10}t^{17} - 247\,s^{11}t^{16} - 390\,s^{12}t^{15} - 474\,s^{13}t^{14} - s^8t^{20} - 13\,s^9t^{19} \\
&-45\,s^{10}t^{18} - 79\,s^{11}t^{17} - 82\,s^{12}t^{16} - 30\,s^{13}t^{15} + 30\,s^{14}t^{14} - 2\,s^9t^{20} + 4\,s^{10}t^{19} \\
&+49\,s^{11}t^{18} + 169\,s^{12}t^{17} + 377\,s^{13}t^{16} + 575\,s^{14}t^{15} + 11\,s^{10}t^{20} + 74\,s^{11}t^{19} \\
&+220\,s^{12}t^{18} + 469\,s^{13}t^{17} + 727\,s^{14}t^{16} + 905\,s^{15}t^{15} + 6\,s^{10}t^{21} + 39\,s^{11}t^{20} \\
&+124\,s^{12}t^{19} + 264\,s^{13}t^{18} + 405\,s^{14}t^{17} + 503\,s^{15}t^{16} + s^{10}t^{22} + 6\,s^{11}t^{21} - s^{12}t^{20} \\
&-26\,s^{13}t^{19} - 124\,s^{14}t^{18} - 262\,s^{15}t^{17} - 399\,s^{16}t^{16} - 5\,s^{11}t^{22} - 51\,s^{12}t^{21} \\
&-193\,s^{13}t^{20} - 478\,s^{14}t^{19} - 874\,s^{15}t^{18} - 1242\,s^{16}t^{17} - 3\,s^{11}t^{23} - 40\,s^{12}t^{22} \\
&-174\,s^{13}t^{21} - 491\,s^{14}t^{20} - 954\,s^{15}t^{19} - 1452\,s^{16}t^{18} - 1750\,s^{17}t^{17} - s^{11}t^{24} \\
&-16\,s^{12}t^{23} - 81\,s^{13}t^{22} - 263\,s^{14}t^{21} - 578\,s^{15}t^{20} - 935\,s^{16}t^{19} - 1205\,s^{17}t^{18} \\
&-3\,s^{12}t^{24} - 10\,s^{13}t^{23} - 33\,s^{14}t^{22} - 74\,s^{15}t^{21} - 118\,s^{16}t^{20} - 106\,s^{17}t^{19} \\
&-77\,s^{18}t^{18} + 10\,s^{13}t^{24} + 59\,s^{14}t^{23} + 201\,s^{15}t^{22} + 459\,s^{16}t^{21} + 793\,s^{17}t^{20} \\
&+1102\,s^{18}t^{19} + 7\,s^{13}t^{25} + 52\,s^{14}t^{24} + 205\,s^{15}t^{23} + 542\,s^{16}t^{22} + 1030\,s^{17}t^{21} \\
&+1515\,s^{18}t^{20} + 1801\,s^{19}t^{19} + 2\,s^{13}t^{26} + 20\,s^{14}t^{25} + 97\,s^{15}t^{24} + 296\,s^{16}t^{23} \\
&+647\,s^{17}t^{22} + 1055\,s^{18}t^{21} + 1346\,s^{19}t^{20} + 3\,s^{14}t^{26} + 15\,s^{15}t^{25} + 52\,s^{16}t^{24} \\
&+113\,s^{17}t^{23} + 208\,s^{18}t^{22} + 272\,s^{19}t^{21} + 319\,s^{20}t^{20} - s^{14}t^{27} - 8\,s^{15}t^{26} \\
&-45\,s^{16}t^{25} - 157\,s^{17}t^{24} - 370\,s^{18}t^{23} - 621\,s^{19}t^{22} - 799\,s^{20}t^{21} - 3\,s^{15}t^{27} \\
&-18\,s^{16}t^{26} - 76\,s^{17}t^{25} - 205\,s^{18}t^{24} - 403\,s^{19}t^{23} - 572\,s^{20}t^{22} - 672\,s^{21}t^{21}.
\end{aligned}
$$

10. Appendix B: Formanek's formulae

Let d be any positive integer. Denote by μ a partition of a positive integer k and by χ^μ the corresponding irreducible complex character of the symmetric group S_k. Define the *length*, $l(\mu)$, of μ to be the number of parts of μ.

Let Λ_d denote the ring of symmetric polynomials in d variables, $t_1, \ldots, t_d$. If μ has at most d parts, we denote by $f_{\mu,d} \in \Lambda_d$ the corresponding Schur function.

Define the Frobenius map Fr_d to be the additive homomorphism from the direct sum of the character rings of all S_k's to Λ_d by setting $\mathrm{Fr}_d(\chi^\mu) = f_{\mu,d}$ for each partition μ of k having at most d parts and $\mathrm{Fr}_d(\chi^\mu) = 0$ otherwise.

We can now state Formanek's formulae for the algebras $C_{n,d}$ and $T_{n,d}$ (see [1]):

$$P(C_{n,d}; t_1, \ldots, t_d) = \sum_{k\geq 0} \mathrm{Fr}_d\left(\theta_n^{(k)}\right)(t_1, \ldots, t_d), \tag{10.1}$$

$$P(T_{n,d}; t_1, \ldots, t_d) = \sum_{k\geq 0} \mathrm{Fr}_d\left(\theta_n^{(k+1)} \downarrow S_k\right)(t_1, \ldots, t_d), \tag{10.2}$$

where $\theta_n^{(k)}$ is a particular character of S_k. This character is defined by the formula

$$\theta_n^{(k)} = \sum_{\mu:\, l(\mu)\leq n} \chi^\mu \otimes \chi^\mu, \tag{10.3}$$

where $\otimes$ is the usual tensor product of characters of S_k.

Recall that we are mainly interested in the case $d = 2$. In that case we set $s = t_1$, $t = t_2$, and $f_\mu = f_{\mu,2}$. If $\mu = (p, q)$ with $p \geq q \geq 0$, then

$$f_\mu = f_{p,q} = (st)^q \left(s^{p-q} + s^{p-q-1}t + \cdots + st^{p-q-1} + t^{p-q}\right). \tag{10.4}$$

If $q = 0$ we shall write $f_p = f_{p,0}$.

By using GAP [11], we computed the first 26 terms of the series (10.1) in the case $(n, d) = (5, 2)$ and we obtained the formula displayed below. By substituting the expressions (10.4) for the Schur functions f_μ into this formula, one obtains an initial chunk of the bivariate Taylor series of $P(C_{5,2}; s, t)$ which agrees with the bivariate Taylor series of the rational function that we have computed by using the Molien–Weyl formula (see Theorem 2.2).

The k-th summand in (10.1), when written as a linear combination of Schur functions, gives the decomposition of the character of the representation of GL_d on the k-th homogeneous component of $C_{n,d}$.

Let $c^\bullet_{n,2}(k)$ denote the sum of the coefficients of the Schur functions $f_{k-p,p}$, $p = 0, 1, \ldots, 2p \leq k$, in (10.1) when $d = 2$. This is the length of the homogeneous component of $C_{n,2}$ of degree k as a GL_2-module. Since $C^\bullet_{n,2}$ is the algebra of GL_1-invariants in $C_{n,2}$, the coefficient of t^{2k} in the Taylor series of $P(C^\bullet_{n,2}; t)$ must be equal to $c^\bullet_{n,2}(2k)$. It is easy to verify this assertion when $n = 5$ and $k \leq 12$ by using the expansion for $P(C_{5,2})$ below and the one for $P(C^\bullet_{5,2})$ at the end of Section 7.

$$
\begin{aligned}
&P(C_{5,2};s,t) = \\
&1 + f_1 + 2\,f_2 + f_{2,1} + 3\,f_3 + 3\,f_{2,2} + 2\,f_{3,1} + 5\,f_4 + 6\,f_{3,2} + 5\,f_{4,1} + 7\,f_5 \\
&+4\,f_{3,3} + 15\,f_{4,2} + 8\,f_{5,1} + 10\,f_6 + 18\,f_{4,3} + 25\,f_{5,2} + 14\,f_{6,1} + 13\,f_7 + 24\,f_{4,4} \\
&+37\,f_{5,3} + 44\,f_{6,2} + 20\,f_{7,1} + 18\,f_8 + 58\,f_{5,4} + 76\,f_{6,3} + 66\,f_{7,2} + 30\,f_{8,1} \\
&+23\,f_9 + 44\,f_{5,5} + 136\,f_{6,4} + 126\,f_{7,3} + 101\,f_{8,2} + 41\,f_{9,1} + 30\,f_{10} + 163\,f_{6,5} \\
&+246\,f_{7,4} + 207\,f_{8,3} + 142\,f_{9,2} + 57\,f_{10,1} + 37\,f_{11} + 171\,f_{6,6} + 354\,f_{7,5} \\
&+431\,f_{8,4} + 311\,f_{9,3} + 200\,f_{10,2} + 74\,f_{11,1} + 47\,f_{12} + 476\,f_{7,6} + 700\,f_{8,5} \\
&+681\,f_{9,4} + 458\,f_{10,3} + 267\,f_{11,2} + 98\,f_{12,1} + 57\,f_{13} + 388\,f_{7,7} + 1080\,f_{8,6} \\
&+1204\,f_{9,5} + 1047\,f_{10,4} + 640\,f_{11,3} + 357\,f_{12,2} + 124\,f_{13,1} + 70\,f_{14} + 1277\,f_{8,7} \\
&+2024\,f_{9,6} + 1973\,f_{10,5} + 1517\,f_{11,4} + 884\,f_{12,3} + 460\,f_{13,2} + 157\,f_{14,1} + 84\,f_{15} \\
&+1166\,f_{8,8} + 2811\,f_{9,7} + 3534\,f_{10,6} + 3014\,f_{11,5} + 2160\,f_{12,4} + 1177\,f_{13,3} \\
&+591\,f_{14,2} + 194\,f_{15,1} + 101\,f_{16} + 3419\,f_{9,8} + 5427\,f_{10,7} + 5682\,f_{11,6} \\
&+4470\,f_{12,5} + 2957\,f_{13,4} + 1550\,f_{14,3} + 740\,f_{15,2} + 240\,f_{16,1} + 119\,f_{17} \\
&+2808\,f_{9,9} + 7592\,f_{10,8} + 9371\,f_{11,7} + 8780\,f_{12,6} + 6352\,f_{13,5} + 3985\,f_{14,4} \\
&+1992\,f_{15,3} + 924\,f_{16,2} + 290\,f_{17,1} + 141\,f_{18} + 8693\,f_{10,9} + 14284\,f_{11,8} \\
&+15284\,f_{12,7} + 12921\,f_{13,6} + 8823\,f_{14,5} + 5232\,f_{15,4} + 2535\,f_{16,3} \\
&+1131\,f_{17,2} + 351\,f_{18,1} + 164\,f_{19} + 7344\,f_{10,10} + 18927\,f_{11,9} + 24781\,f_{12,8} \\
&+23520\,f_{13,7} + 18489\,f_{14,6} + 11909\,f_{15,5} + 6784\,f_{16,4} + 3167\,f_{17,3} + 1380\,f_{18,2} \\
&+417\,f_{19,1} + 192\,f_{20} + 21565\,f_{11,10} + 35929\,f_{12,9} + 40009\,f_{13,8} + 34897\,f_{14,7} \\
&+25625\,f_{15,6} + 15798\,f_{16,5} + 8622\,f_{17,4} + 3926\,f_{18,3} + 1658\,f_{19,2} + 496\,f_{20,1} \\
&+221\,f_{21} + 17281\,f_{11,11} + 46991\,f_{12,10} + 61801\,f_{13,9} + 61722\,f_{14,8} \\
&+49917\,f_{15,7} + 34778\,f_{16,6} + 20520\,f_{17,5} + 10849\,f_{18,4} + 4796\,f_{19,3} + 1986\,f_{20,2} \\
&+582\,f_{21,1} + 255\,f_{22} + 51694\,f_{12,11} + 87853\,f_{13,10} + 100058\,f_{14,9} + 91235\,f_{15,8} \\
&+69582\,f_{16,7} + 46117\,f_{17,6} + 26294\,f_{18,5} + 13444\,f_{19,4} + 5820\,f_{20,3} + 2350\,f_{21,2} \\
&+682\,f_{22,1} + 291\,f_{23} + 41569\,f_{12,12} + 111058\,f_{13,11} + 150865\,f_{14,10} \\
&+153818\,f_{15,9} + 130796\,f_{16,8} + 94507\,f_{17,7} + 60179\,f_{18,6} + 33154\,f_{19,5} \\
&+16519\,f_{20,4} + 6983\,f_{21,3} + 2772\,f_{22,2} + 790\,f_{23,1} + 333\,f_{24} + 120672\,f_{13,12} \\
&+207439\,f_{14,11} + 242629\,f_{15,10} + 227776\,f_{16,9} + 182080\,f_{17,8} + 125907\,f_{18,7} \\
&+77153\,f_{19,6} + 41349\,f_{20,5} + 20055\,f_{21,4} + 8328\,f_{22,3} + 3237\,f_{23,2} \\
&+915\,f_{24,1} + 377\,f_{25} + \cdots .
\end{aligned}
$$

11. Appendix C: Taylor expansions of $P(C_{n,2};t)$ and $P(C_{n,2}(0);t)$

In Table 9 we give the Taylor series of $P(C_{n,2};t)$ and $P(C_{n,2}(0);t)$ for $n \le 6$ including the terms of degree $k < 20$. In the former case we also tabulate (see Table 10) the coefficients for $k \le 12$ and make a couple of observations. We are then lead to some speculations concerning certain limits of the algebras $C_{n,d}$, which we are going to introduce now.

Table 9: Taylor expansions of $P(C_{n,2};t)$ and $P(C_{n,2}(0);t)$

$$
\begin{aligned}
P(C_{0,2};t) &= 1,\\
P(C_{1,2};t) &= 1+2\,t+3\,t^2+4\,t^3+5\,t^4+6\,t^5+7\,t^6+8\,t^7+9\,t^8+10\,t^9\\
&\quad +11\,t^{10}+12\,t^{11}+13\,t^{12}+14\,t^{13}+15\,t^{14}+16\,t^{15}+17\,t^{16}\\
&\quad +18\,t^{17}+19\,t^{18}+20\,t^{19}+\cdots,\\
P(C_{1,2}(0);t) &= 1,\\
P(C_{2,2};t) &= 1+2\,t+6\,t^2+10\,t^3+20\,t^4+30\,t^5+50\,t^6+70\,t^7+105\,t^8\\
&\quad +140\,t^9+196\,t^{10}+252\,t^{11}+336\,t^{12}+420\,t^{13}+540\,t^{14}+660\,t^{15}\\
&\quad +825\,t^{16}+990\,t^{17}+1210\,t^{18}+1430\,t^{19}+\cdots,\\
P(C_{2,2}(0);t) &= 1+3\,t^2+6\,t^4+10\,t^6+15\,t^8+21\,t^{10}+28\,t^{12}+36\,t^{14}+45\,t^{16}\\
&\quad +55\,t^{18}+\cdots,\\
P(C_{3,2};t) &= 1+2\,t+6\,t^2+14\,t^3+29\,t^4+56\,t^5+107\,t^6+186\,t^7+320\,t^8\\
&\quad +530\,t^9+851\,t^{10}+1332\,t^{11}+2051\,t^{12}+3074\,t^{13}+4544\,t^{14}\\
&\quad +6602\,t^{15}+9444\,t^{16}+13322\,t^{17}+18579\,t^{18}+25564\,t^{19}+\cdots,\\
P(C_{3,2}(0);t) &= 1+3\,t^2+4\,t^3+7\,t^4+12\,t^5+24\,t^6+28\,t^7+55\,t^8+76\,t^9\\
&\quad +111\,t^{10}+160\,t^{11}+238\,t^{12}+304\,t^{13}+447\,t^{14}+588\,t^{15}\\
&\quad +784\,t^{16}+1036\,t^{17}+1379\,t^{18}+1728\,t^{19}+\cdots,\\
P(C_{4,2};t) &= 1+2\,t+6\,t^2+14\,t^3+34\,t^4+68\,t^5+144\,t^6+276\,t^7+534\,t^8\\
&\quad +974\,t^9+1774\,t^{10}+3106\,t^{11}+5410\,t^{12}+9146\,t^{13}+15334\,t^{14}\\
&\quad +25158\,t^{15}+40884\,t^{16}+65264\,t^{17}+103204\,t^{18}\\
&\quad +160808\,t^{19}+\cdots,\\
P(C_{4,2}(0);t) &= 1+3\,t^2+4\,t^3+12\,t^4+14\,t^5+42\,t^6+56\,t^7+126\,t^8+182\,t^9\\
&\quad +360\,t^{10}+532\,t^{11}+972\,t^{12}+1432\,t^{13}+2452\,t^{14}+3636\,t^{15}\\
&\quad +5902\,t^{16}+8654\,t^{17}+13560\,t^{18}+19664\,t^{19}+\cdots,\\
P(C_{5,2};t) &= 1+2\,t+6\,t^2+14\,t^3+34\,t^4+74\,t^5+159\,t^6+324\,t^7+657\,t^8\\
&\quad +1286\,t^9+2488\,t^{10}+4702\,t^{11}+8790\,t^{12}+16146\,t^{13}\\
&\quad +29326\,t^{14}+52526\,t^{15}+93064\,t^{16}+162910\,t^{17}+282267\,t^{18}\\
&\quad +483792\,t^{19}+\cdots,\\
P(C_{5,2}(0);t) &= 1+3\,t^2+4\,t^3+12\,t^4+20\,t^5+45\,t^6+80\,t^7+168\,t^8+296\,t^9\\
&\quad +573\,t^{10}+1012\,t^{11}+1874\,t^{12}+3268\,t^{13}+5824\,t^{14}+10020\,t^{15}\\
&\quad +17338\,t^{16}+29308\,t^{17}+49511\,t^{18}+82168\,t^{19}+\cdots,\\
P(C_{6,2};t) &= 1+2\,t+6\,t^2+14\,t^3+34\,t^4+74\,t^5+166\,t^6+342\,t^7+716\,t^8\\
&\quad +1442\,t^9+2898\,t^{10}+5686\,t^{11}+11122\,t^{12}+21366\,t^{13}\\
&\quad +40842\,t^{14}+77098\,t^{15}+144581\,t^{16}+268376\,t^{17}+494812\,t^{18}\\
&\quad +904056\,t^{19}+\cdots,\\
P(C_{6,2}(0);t) &= 1+3\,t^2+4\,t^3+12\,t^4+20\,t^5+52\,t^6+84\,t^7+198\,t^8+352\,t^9\\
&\quad +730\,t^{10}+1332\,t^{11}+2648\,t^{12}+4808\,t^{13}+9232\,t^{14}+16780\,t^{15}\\
&\quad +31227\,t^{16}+56312\,t^{17}+102641\,t^{18}+182808\,t^{19}+\cdots.
\end{aligned}
$$

Define the $\mathbf{Z}^d$-graded algebra $C_{\infty,d}$ as the inverse limit of

$$C_{1,d} \leftarrow C_{2,d} \leftarrow C_{3,d} \leftarrow \cdots .$$

By adapting a definition of Formanek [8, p. 52], we refer to $C_{\infty,d}$ as the *pure free trace ring* on d generators. One can next take the direct limit of

$$C_{\infty,1} \rightarrow C_{\infty,2} \rightarrow C_{\infty,3} \rightarrow \cdots$$

to obtain the $\mathbf{Z}^\infty$-graded algebra $C_{\infty,\infty}$, which is the pure free trace ring on countably many generators $x_1, x_2, x_3, \ldots$. It follows from the Second Fundamental Theorem for invariants of $n \times n$ matrices (see e.g. [8, Theorem 50]) that $C_{\infty,\infty}$ is indeed isomorphic to the pure free trace ring as defined by Formanek.

Let us write $c_{n,2}(k)$ for the coefficients of the Poincaré series

$$P(C_{n,2}; t) = \sum_{k \geq 0} c_{n,2}(k) t^k$$

and let us display these coefficients in an infinite table (rows indexed by $n \geq 0$ and columns by $k \geq 0$). The data from Table 9 give the top portion of Table 10.

Table 10: The coefficients $c_{n,2}(k)$

1	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	2	6	10	20	30	50	70	105	140	196	252	336	
1	2	6	14	29	56	107	186	320	530	851	1332	2051	
1	2	6	14	34	68	144	276	534	974	1774	3106	5410	
1	2	6	14	34	74	159	324	657	1286	2488	4702	8790	
1	2	6	14	34	74	166	342	716	1442	2898	5686	11122	
1	2	6	14	34	74	166	350	737	1512	3087	6194	12376	
1	2	6	14	34	74	166	350	746	1536	3168	6416	12982	
1	2	6	14	34	74	166	350	746	1546	3195	6508	13237	
1	2	6	14	34	74	166	350	746	1546	3206	6538	13340	
1	2	6	14	34	74	166	350	746	1546	3206	6550	13373	
1	2	6	14	34	74	166	350	746	1546	3206	6550	13386	
$\vdots$													

We observed from the top part of this table that apparently each column stabilizes and that

$$\lim_{n \to \infty} P(C_{n,2}; t) = \sum_{k \geq 0} c_{n,2}(n) t^n.$$

After making this observation, we looked up the diagonal sequence

$$\{c_{n,2}(n)\}_{n \geq 0} = 1, 2, 6, 14, 34, 74, 166, \ldots$$

in the On-Line Encyclopedia of Integer Sequences [19] by entering only these 7 integers. We were pleasantly surprised to find that it is registered there as the sequence A070933, and identified as the sequence of coefficients in the power series expansion of the infinite product

$$\prod_{k \geq 1} \frac{1}{1 - 2t^k}.$$

The first 30 terms of A070933 are listed in [19]. The above infinite product should be the Poincaré series of the algebra $C_{\infty,2}$ (see the definition below). In the bigraded case one would have to replace the above product with

$$\prod_{k\geq 1} \frac{1}{1-s^k-t^k}.$$

More generally, we expect that the multigraded Poincaré series of $C_{\infty,d}$ and $C_{\infty,\infty}$ be given by

$$P(C_{\infty,d};t_1,\dots,t_d) = \prod_{k\geq 1} \frac{1}{1-t_1^k-\cdots-t_d^k}$$

and

$$P(C_{\infty,\infty};t_1,t_2,\dots) = \prod_{k\geq 1} \frac{1}{1-p_k}, \tag{11.1}$$

respectively, where the p_k are the usual power sum symmetric functions:

$$p_k = t_1^k + t_2^k + \cdots .$$

The latter formula is indeed valid. As explained in [7], it follows from the Procesi–Razmyslov theorem that

$$P(C_{n,\infty};t) = \sum_{\mu;\, l(\mu)\leq n} \mathrm{Fr}(\chi^\mu \otimes \chi^\mu).$$

The Frobenius map Fr is the additive map from the direct sum of the character rings of all S_k's to the ring, Λ, of symmetric functions in infinitely many variables $t_1, t_2, \dots$. It is defined by setting, for all partitions μ, $\mathrm{Fr}(\chi^\mu) = f_\mu \in \Lambda$, the Schur function corresponding to the partition μ. By letting $n \to \infty$, we obtain that

$$P(C_{\infty,\infty};t) = \sum_{\mu} \mathrm{Fr}(\chi^\mu \otimes \chi^\mu),$$

where the summation is now over all partitions μ. It remains to observe that the right hand side of this formula and the one of (11.1) are equal, see Macdonald's classic [12, Chapter 1, Section 7, Example 9(a)].

Another interesting observation about the above table is that apparently the second differences

$$\alpha_k = c_{n,2}(n+k) - c_{n-1,2}(n+k) - c_{n-1,2}(n+k-1) + c_{n-2,2}(n+k-1)$$

are independent of n for $n \geq k$. The sequence

$$\{\alpha_k\}_{k\geq 0} = 1,\ 3,\ 11,\ 33,\ 98,\ 270,\ \dots \tag{11.2}$$

has not been recorded so far in [19]. By using the sequence A070933 and the hypothetical rules mentioned above, we extended the top portion of Table 10 with 6 additional rows. (Actually the above rules allow us to fill in the next three columns of the table, but the page is not wide enough.) Subsequently, we verified the validity of these additional rows by using Formanek's formula, and at the same time we enlarged the number of columns to 26, i.e., $0 \leq k \leq 25$. This made it possible to compute a few more terms of the sequence (11.2):

$$1,\ 3,\ 11,\ 33,\ 98,\ 270,\ 736,\ 1932,\ 5009,\ 12727,\ 31977,\ 79307,\ 194947,\ \dots$$

This sequence has another hypothetical incarnation. Recall the integers $c^\bullet_{n,2}(k)$ introduced in Appendix B. We conjecture that, for each fixed $k \ge 0$,

$$\alpha_k = c^\bullet_{n,2}(n+k) - c^\bullet_{n-1,2}(n+k)$$

is valid for sufficiently large n.

12. Appendix D: Taylor expansion of $P(T_{n,2};t)$

In this appendix we give, for $n \le 6$, the coefficients of t^k, $k < 20$, in the power series expansions of $P(T_{n,2};t)$. We also tabulate the coefficients for $k \le 12$ and make some interesting observations about the table.

Table 11: Taylor expansions of $P(T_{n,2};t)$

$$\begin{aligned}
P(T_{0,2};t) &= 1,\\
P(T_{1,2};t) &= 1+2\,t+3\,t^2+4\,t^3+5\,t^4+6\,t^5+7\,t^6+8\,t^7+9\,t^8+10\,t^9\\
&\quad +11\,t^{10}+12\,t^{11}+13\,t^{12}+14\,t^{13}+15\,t^{14}+16\,t^{15}+17\,t^{16}\\
&\quad +18\,t^{17}+19\,t^{18}+20\,t^{19}+\cdots,\\
P(T_{2,2};t) &= 1+4\,t+11\,t^2+24\,t^3+46\,t^4+80\,t^5+130\,t^6+200\,t^7+295\,t^8\\
&\quad +420\,t^9+581\,t^{10}+784\,t^{11}+1036\,t^{12}+1344\,t^{13}+1716\,t^{14}\\
&\quad +2160\,t^{15}+2685\,t^{16}+3300\,t^{17}+4015\,t^{18}+4840\,t^{19}+\cdots,\\
P(T_{3,2};t) &= 1+4\,t+14\,t^2+38\,t^3+93\,t^4+204\,t^5+419\,t^6+806\,t^7+1480\,t^8\\
&\quad +2600\,t^9+4411\,t^{10}+7244\,t^{11}+11579\,t^{12}+18048\,t^{13}+27528\,t^{14}\\
&\quad +41150\,t^{15}+60428\,t^{16}+87280\,t^{17}+124203\,t^{18}+174308\,t^{19}+\cdots,\\
P(T_{4,2};t) &= 1+4\,t+14\,t^2+42\,t^3+113\,t^4+278\,t^5+646\,t^6+1418\,t^7+2979\,t^8\\
&\quad +6018\,t^9+11752\,t^{10}+22256\,t^{11}+41030\,t^{12}+73784\,t^{13}\\
&\quad +129748\,t^{14}+223498\,t^{15}+377753\,t^{16}+627314\,t^{17}\\
&\quad +1024882\,t^{18}+1649026\,t^{19}+\cdots,\\
P(T_{5,2};t) &= 1+4\,t+14\,t^2+42\,t^3+118\,t^4+304\,t^5+747\,t^6+1748\,t^7+3949\,t^8\\
&\quad +8620\,t^9+18296\,t^{10}+37818\,t^{11}+76398\,t^{12}+151022\,t^{13}\\
&\quad +292754\,t^{14}+557130\,t^{15}+1042364\,t^{16}+1919044\,t^{17}\\
&\quad +3480203\,t^{18}+6221668\,t^{19}+\cdots,\\
P(T_{6,2};t) &= 1+4\,t+14\,t^2+42\,t^3+118\,t^4+310\,t^5+779\,t^6+1876\,t^7+4382\,t^8\\
&\quad +9948\,t^9+22057\,t^{10}+47850\,t^{11}+101844\,t^{12}+212946\,t^{13}\\
&\quad +438118\,t^{14}+887814\,t^{15}+1773827\,t^{16}+3496850\,t^{17}\\
&\quad +6806682\,t^{18}+13089748\,t^{19}+\cdots.
\end{aligned}$$

Let us write $d_{n,2}(k)$ for the coefficients of the Poincaré series

$$P(T_{n,2};t) = \sum_{k\ge 0} d_{n,2}(k)t^k$$

and let us display these coefficients in an infinite table (rows indexed by $n \ge 0$ and columns by $k \ge 0$).

Table 12: The coefficients $d_{n,2}(k)$

1	0	0	0	0	0	0	0	0	0	0	0	0	$\cdots$
1	2	3	4	5	6	7	8	9	10	11	12	13	
1	4	11	24	46	80	130	200	295	420	581	784	1036	
1	4	14	38	93	204	419	806	1480	2600	4411	7244	11579	
1	4	14	42	113	278	646	1418	2979	6018	11752	22256	41030	
1	4	14	42	118	304	747	1748	3949	8620	18296	37818	76398	
1	4	14	42	118	310	779	1876	4382	9948	22057	47850	101844	
$\vdots$													

By taking first the inverse limit of

$$T_{1,d} \leftarrow T_{2,d} \leftarrow T_{3,d} \leftarrow \cdots,$$

one obtains the $\mathbf{Z}^d$-graded algebra $T_{\infty,d}$. By adapting a definition of Formanek [8, p. 52], we refer to $T_{\infty,d}$ as the *mixed free trace ring* on d generators. One can next take the direct limit of

$$T_{\infty,1} \to T_{\infty,2} \to T_{\infty,3} \to \cdots$$

to obtain the $\mathbf{Z}^\infty$-graded algebra $T_{\infty,\infty}$, which is the mixed free trace ring on countably many generators $x_1, x_2, x_3, \ldots$.

There is a close relationship between Tables 10 and 12 from which we deduce that the following formula apparently holds:

$$P(C_{\infty,2};t) = (1-2t)P(T_{\infty,2};t).$$

By further heuristic reasoning, one obtains the hypothetical formulae

$$P(T_{\infty,d};t_1,\ldots,t_d) = \frac{1}{(1-t_1-\cdots-t_d)^2}\prod_{k\geq 2}\frac{1}{1-t_1^k-\cdots-t_d^k}$$

and

$$P(T_{\infty,\infty};t_1,t_2,\ldots) = \frac{1}{(1-p_1)^2}\prod_{k\geq 2}\frac{1}{1-p_k}.$$

Similarly as in the previous appendix, the second differences of the coefficients $d_{n,2}(n+k)$ provide yet another sequence:

$$1,\ 6,\ 27,\ 103,\ 358,\ 1159,\ \ldots$$

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
E-mail address: djokovic@uwaterloo.ca